Deferred Correction

Constructing an $\mathcal{O}(h^4)$ approximation for Poisson's equation from the difference equation 5-point approximation.

Consider the 5-point Laplace fin. difference approxim. (uniform grid)

$$\boxed{\nabla_5^2 v_{ij} = \frac{1}{h^2}\left[v_{i,j-1} + v_{i-1,j} - 4v_{ij} + v_{i+1,j} + v_{i,j+1}\right] = 0} \quad (6.1)$$

$i,j = 1 \ldots m.$

Substituting a solution, $u(x,y)$, of Laplace equation $\nabla^2 u = 0$ into (6.1) leads to

$$\nabla_5^2 u_{ij} = \frac{1}{h^2}\left[u_{i,j-1} + u_{i-1,j} - 4u_{ij} + u_{i+1,j} + u_{i,j+1}\right] =$$

$$\underset{\text{series}}{\overset{\text{Expanding in Taylor}}{=}} \nabla^2 u_{ij} + \frac{h^2}{12}\left[(u_{xxxx})_{ij} + (u_{yyyy})_{ij}\right] + \mathcal{O}(h^4)$$

Therefore,

Local Truncation error $\boxed{(\tau_5)_{ij} = \nabla_5^2 u_{ij} - \nabla^2 u_{ij}} = \frac{h^2}{12}\left[(u_{4x})_{ij} + (u_{4y})_{ij}\right] + \mathcal{O}(h^4)$

or

$$\boxed{(\tau_5)_{ij} = \mathcal{O}(h^2).}$$

Conclusion:

The discrete operator $\nabla_5^2 U_{ij}$ defined by (6.1) is an $\mathcal{O}(h^2)$ approximation to $\nabla^2 u(x_i, y_j)$ sufficiently smooth functions $u(x,y)$.

Goal:

Define a new FD approx. $\tilde{\nabla}_5^2 \tilde{U}_{ij} = 0$ that approximates $\nabla^2 u(x_i, y_j) \overset{=0}{}$ to $\mathcal{O}(h^4)$.

Idea:

Notice that for u sufficiently smooth

$$\boxed{\left(\nabla_5^2 u_{ij} - \frac{h^2}{12}\left[(u_{4x})_{ij} + (u_{4y})_{ij}\right]\right) - \left(\nabla^2 u_{ij}\right) = \mathcal{O}(h^4)}$$

So by incorporating the leading order term of truncation error into our finite difference approximation, it might be possible to obtain a new 4th order F.D. approx.

The previous statement suggests the definition of the discrete operator

$$\boxed{\tilde{\nabla}_5^2 \tilde{U}_{ij} \stackrel{\text{def}}{=} \nabla_5^2 \tilde{U}_{i,j} - \frac{h^2}{12}\left[D_{4x} U_{ij} + D_{4y} U_{ij} \right]} \qquad (8.1)$$

where $\underline{U_{ij} \text{ is a}}$ $\underline{2^{\text{nd}} \text{ order approx. of } u(x_i, y_j)}$.

which may be a discrete solution of the 5-point stencil approx. of Laplace's equ.

$$\nabla_5^2 U_{ij} = 0.$$

and $D_{4x} V_{ij}$ and $D_{4y} W_{ij}$ are Second order $\mathcal{O}(h^2)$

Approximations of $u_{4x}(x_i, y_j)$ and $u_{4y}(x_i, y_j)$, respectively.

<u>Theorem 2.</u> The Finite Difference scheme $\tilde{\nabla}_2^5 \tilde{U}_{ij} = 0$ gives

$\mathcal{O}(h^4)$ approximation of $\nabla^2 u(x_i, y_j) = 0$

Before attempting a proof. we will prove the following Lemma (Homework problem).

$$D_{4x} U_{ij} \stackrel{\text{def}}{\equiv} \frac{1}{h^4}\left[U_{i-2,j} - 4U_{i-1,j} + 6U_{ij} - 4U_{i+1,j} + U_{i+2,j} \right]$$

u Suff. Smooth,

$$D_{4x} u_{ij} = \frac{1}{h^4}\left[u_{i-2,j} - 4u_{i-1,j} + 6u_{ij} - 4u_{i+1,j} + u_{i+2,j} \right] = (u_{4x})_{ij} + \frac{h^2}{6}(u_{6x})_{ij} + \mathcal{O}(h^4).$$

<u>Lemma.</u> Assume there exist sufficiently differentiable functions $\alpha(x,y)$, $\beta_u(x,y)$ and $\gamma_u(x,y)$ such that

① $U_{ij} = u(x_i, y_j) + h^2 \alpha(x_i, y_j) + O(h^3)$.

② $D^2_{4x} U_{ij} = u_{4x}(x_i, y_j) + h^2 \beta_u(x_i, y_j) + O(h^3)$

③ $D^2_{4y} U_{ij} = u_{4y}(x_i, y_j) + h^2 \gamma_u(x_i, y_j) + O(h^3)$

Then,

a) $D_{4x} U_{ij}$ is also $O(h^2)$ approx. of $(u_{4x})(x_i, y_j)$

b) $D_{4y} U_{ij}$ " " " " " $(u_{4y})(x_i, y_j)$.

Proof. (Homework problem).

Proof (Theorem 2 page 8)

Substituting a solution $U(x,y)$ of $\nabla^2 u = 0$ into (8.1). It means replacing $\tilde{v}_{ij}$ by $u_{ij} = U(x_i, y_j)$ leads to

$$\tilde{\nabla}_5^2 u_{ij} = \nabla_5^2 u_{ij} - \frac{h^2}{12}\left[D_{4x} v_{ij} + D_{4y} v_{ij}\right]$$

Thus, using the definition of $\nabla_5^2(\cdot)$

$$\tilde{\nabla}_5^2 u_{ij} = \nabla^2 u_{ij} + \frac{h^2}{12}\left[(u_{4x})_{ij} + (u_{4y})_{ij}\right] + O(h^4)$$
$$- \frac{h^2}{12}\left[D_{4x} v_{ij} + D_{4y} v_{ij}\right]$$

$$= \nabla^2 u_{ij} + \frac{h^2}{12}\left[(u_{4x})_{ij} - D_{4x} v_{ij}\right]$$
$$+ \frac{h^2}{12}\left[(u_{4y})_{ij} - D_{4y} v_{ij}\right] + O(h^4)$$

Therefore, using Lemma

$$\tilde{\nabla}_5^2 u_{ij} = \nabla^2 u_{ij} + \frac{h^2}{12}\left(O(h^2)\right) + \frac{h^2}{12}\left(O(h^2)\right) + O(h^4)$$

$$\Rightarrow \quad \boxed{\tilde{\nabla}_5^2 u_{ij} - \nabla^2 u_{ij} = O(h^4).}$$

If $U(x,y)$ is a true solution of $\nabla^2 u = 0$, then

$$\text{L.T.E} = \tau_{ij} = \tilde{\nabla}_5^2 u_{ij} = O(h^4).$$

local truncation error

Corollary 1.

The local truncation error of the finite difference approximation of Poisson's equation given by

$$\tilde{\nabla}_5^2 \tilde{U}_{ij} = f_{ij}, \qquad i,j=1,..,m \qquad (11.1)$$

is $\mathcal{O}(h^4)$.

Proof. Consider Poisson's equation

$$\nabla^2 u(x,y) = f(x,y) \qquad (11.2)$$

If a solution $u(x,y)$ of (11.2) is substituted into (11.1), then

$$\tilde{\nabla}_5^2 u_{ij} - f_{ij} = \nabla_2 u_{ij} + \mathcal{O}(h^4) - f_{ij}$$

Thus,

$$\tilde{\nabla}_5^2 u_{ij} - f_{ij} - \cancelto{0}{(\nabla^2 u_{ij} - f_{ij})} = \mathcal{O}(h^4)$$

or equivalently

$$(\tilde{\tau}_5)_{ij} = \tilde{\nabla}_5^2 u_{ij} - f_{ij} - \cancelto{0}{(\nabla^2 u_{ij} - f_{ij})} = \mathcal{O}(h^4)$$

Recall, $\vec{\tau} = A\hat{U} - \vec{F}$, where $\hat{U} = u$ true solution.