2.13 Existence and Uniqueness

Case 1.- Nohomogeneous Dirichlet Problem.

$$\begin{cases} u''(x) = f(x), & 0<x<1 \quad (1.1) \\ u(0)=\alpha, \quad u(1)=\beta & \quad (1.2) \end{cases}$$

Solution: Assuming $G'(x) = \int f(x)$

then, $u(x) = G(x) + C_1 x + C_2$ (integrating (1.1) twice)

Using the BCs

$$\alpha = u(0) = G(0) + C_2 \Rightarrow C_2 = \alpha - G(0)$$

$$\beta = u(1) = G(1) + C_1 + \alpha - G(0)$$

$$\Rightarrow C_1 = \beta - G(1) - \alpha + G(0)$$

and as consequence,

$$\boxed{u(x) = G(x) + [\beta - \alpha + G(0) - G(1)]\,x + \alpha - G(0).} \quad (1.3)$$

Is this Solution unique?

Consider two possible solutions: $u_1(x)$ and $u_2(x)$

then, $\omega(x) \equiv u_1(x) - u_2(x)$ Satisfies

$$\begin{cases} \omega''(x) = 0, & 0<x<1 \quad (1.4) \\ \omega(0) = 0, \quad \omega(1) = 0 & \quad (1.5) \end{cases}$$

Remark: From Physical pples, (1.1) - (1.2) always has a soln because an equillibrium can be reached whenever at least in one end the flux is not prescribed.

Temperature is fixed at the ends but flux is not

Any excess of thermal energy (source) generated inside rod will be going out at the ends so temp. at ends remains unchanged.

Same if heat is absorbed inside bar (Sink).
In this case, the flow will be directed inside the bar.

It can be shown that the only Solution of this BVP is the trivial solution: $w(x) \equiv 0$, for all $x \in [a,b]$.

This is inmediately seen by direct integration.

Then,

$$u_1(x) = u_2(x)$$

and BVP (1.1)-(1.2) has a unique Soln.

Compare this results for the homogeneous and nonhomogeneous BVPs with the following theorem for Linear Systems of algebraic equations. for a matrix $A_{n\times n}$.

$A\vec{x}=\vec{0}$ has only the trivial Solution $\iff$ Nohomogeneous System $A\vec{x}=\vec{b}$ has a unique Solution for any $\vec{b}$.

Consider

Case 2.- Nonhomogeneous Neumann Problem

$$\begin{cases} u''(x) = f(x), & 0<x<1 \qquad (2.1) \\ u'(0) = \sigma_0, \quad u'(1) = \sigma_1 & \qquad (2.2) \end{cases}$$

Discuss existence of solutions from physical pples. first

Consider the two cases: $\sigma_0, \sigma_1 \neq 0$ and $\sigma_1 = \sigma_0 = 0$.

By integrating,

$$u(x) = G(x) + C_1 x + C_2 \, , \qquad \text{where} \quad G'(x) = \int f(x)\,dx$$

Using BCs:

$$u'(x) = G'(x) + C_1 \, , \text{ then}$$

$$\sigma_0 = u'(0) = G'(0) + C_1 \Rightarrow C_1 = \sigma_0 - G'(0)$$

$$\sigma_1 = u'(1) = G'(1) + C_1 \Rightarrow C_1 = \sigma_1 - G'(1)$$

So the only possibility for existence of solns is that

$$\boxed{G'(1) - G'(0) = \sigma_1 - \sigma_0.} \qquad (3.1)$$

Which is equivalent to

$$\boxed{\int_0^1 f(x)\,dx = \sigma_1 - \sigma_0} \qquad (3.2)$$

Physical Interpretation?

Since $\overset{\sigma_1}{\overset{\shortparallel}{u'(1)}} - \overset{\sigma_0}{\overset{\shortparallel}{u'(0)}} = \int_0^1 u''(x)\,dx = \int_0^1 f(x)\,dx.$

by integration of (2.1) and using (2.2)

Condition (3.2) is called a compatibility condition.

In general, (3.2) is not satisfied because σ_1, σ_0 and $f(x)$ are arbitrary & independent of each other.

In the special case that (3.2) is verified, the Soln. is not unique because C_2 is undetermined. In fact,

$$\boxed{u(x) = G(x) + (G'(0) - \sigma_0)\,x + C_2} \qquad (4.1)$$

What can you say about the corresponding homogeneous problem

$$\begin{cases} u''(x) = 0, & 0 < x < 1 \qquad (4.2) \\ u'(0) = 0, & u'(1) = 0 \quad ? \qquad (4.3) \end{cases}$$

Compare with the relationship between the linear system of algebraic equations with singular matrix A,

$$A\vec{x} = \vec{b} \quad \text{and} \quad A\vec{x} = \vec{0}.$$

How can the unknown constant C_2 be calculated from I.C's of the original Heat cond. BVP ?

<u>Special case:</u> $\sigma_1 = \sigma_2 = 0$ and $f(x) \equiv 0$.

$$\begin{cases} u''(x) = 0 \\ u'(0) = 0, \quad u'(1) = 0 \end{cases}$$

$$u(x) \equiv C_2 \quad \underset{\text{Soln.}}{\text{Equilibrium}} \quad \text{Infinitely many.}$$

Now, C_2 can be calculated from ICs of original heat conduction problem.

In fact, if soln. of PDE: $\bar{u}(x,t)$

and there is an equilibrium solution

$$\frac{d}{dt} E(t) = \frac{d}{dt} \int_0^1 c\rho \, \bar{u}(x,t) \, dx = 0$$

$$\Rightarrow \quad E(\infty) = E(0)$$

or $$\int_0^1 c\rho \, \bar{u}(x,\infty) \, dx = \int_0^1 c\rho \, \bar{u}(x,0) \, dx$$

or $$c\rho \int_0^1 \overset{= C_2}{u(x)} \, dx = c\rho \int_0^1 \overset{\text{I.C.}}{u_0(x)} \, dx$$

$$\Rightarrow \quad C_2 = \int_0^1 u_0(x) \, dx. \qquad (\text{AVG. Temp. in } [0,1]).$$

Determination of c_2 in (4.1). General case

We go back to the conservation of energy equation:

$$\frac{dE}{dt}(t) = \frac{d}{dt}\left[\cancel{A}\int_0^1 c\rho\,\bar{u}(x,t)\,dx\right] = \phi(0,t)\cancel{A} - \phi(1,t)\cancel{A} + \cancel{A}\int_0^1 F(x)\,dx$$

Where $\boxed{-K_0 F(x) = f(x)}$

$$= -K_0 \underbrace{\frac{\partial \bar{u}}{\partial x}(0,t)}_{=\sigma_0} + K_0 \underbrace{\frac{\partial \bar{u}}{\partial x}(1,t)}_{=\sigma_1} + \int_0^1 F(x)\,dx.$$

Now, if there is equillibrium, then

$$\frac{dE}{dt}(t) = 0 \Rightarrow E(t) \equiv \text{constant}$$

$$\Rightarrow \quad \sigma_1 - \sigma_0 = -\frac{1}{K_0}\int_0^1 F(x)\,dx = \int_0^1 f(x)\,dx$$

Same as (3.2)

But also, $E(t_\infty) = E(0)$ ($t=0$)

$$\Rightarrow \quad \int_0^1 c\rho\,\bar{u}(x,t_\infty)\,dx = \int_0^1 c\rho\,\bar{u}(x,0)\,dx = \int_0^1 u_0(x)\,dx$$

or $$\int_0^1 u(x)\,dx = \int_0^1 u_0(x)\,dx$$

or $$\int_0^1 \left\{\left[G(x) + (G'(0) - \sigma_0)x\right] + c_2\right\}dx = \int_0^1 u_0(x)\,dx$$

$$\Rightarrow \quad \boxed{c_2 = \int_0^1 u_0(x)\,dx - \int_0^1 \left[G(x) + (G'(0) - \sigma_0)x\right]dx} \quad (5.1)$$

Consider the FD approx. of (2.1)-(2.2).
using centered difference at the two ends.

$$
\underbrace{\frac{1}{h^2}\begin{bmatrix} -h & h & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & & & 0 \\ \vdots & & & 1 & -2 & 1 \\ 0 & \cdots & & 0 & h & -h \end{bmatrix}}_{A}
\overbrace{\begin{bmatrix} U_0 \\ U_1 \\ \vdots \\ \\ U_m \\ U_{m+1} \end{bmatrix}}^{\vec{U}}
=
\overbrace{\begin{bmatrix} \sigma_0 + \frac{h}{2} f(x_0) \\ f(x_1) \\ \vdots \\ \\ f(x_m) \\ -\sigma_1 + \frac{h}{2} f(x_{m+1}) \end{bmatrix}}^{\vec{F}}
$$

Singular matrix !!

It can be shown (Homework 2) that this system has infinitely many solutions if condition

$$
\underbrace{\boxed{\frac{h}{2} f(x_0) + h \sum_{i=1}^{m} f(x_i) + \frac{h}{2} f(x_{m+1}) = \sigma_1 - \sigma_0}}
$$

which is trapezoidal rule approximation of (3.2).