Finite Element Method for BVPs (1D).

BVP in Strong form

$$\boxed{\begin{cases} -\dfrac{d}{dx}\left(K(x)\dfrac{du}{dx}\right) = f(x), & 0<x<\ell. \quad (1.1)\\ u(0)=0, \quad u(\ell)=0 & \quad (1.2)\end{cases}}$$

Weak-form of (1.1)–(1.2):

Find $u \in V$ such that

$$a(u,v) = (f,v), \quad \text{for all } v \in V \qquad (1.3)$$

where

$$a(u,v) \stackrel{def}{=} \int_0^\ell K(x)\, u'(x)\, v'(x)\, dx$$

$$(f,v) \stackrel{def}{=} \int_0^\ell f(x)\, v(x)\, dx.$$

but V is actually more general than C_D^2 (*)

If $V = C_D^2[0,\ell]$, it can be proved that (1.1)–(1.2) is equivalent to (1.3). (Homework probl.)

The weak-form (1.3) is obtained from (1.1), multiplying it by a "test function" $v \in V$ and integrating. Integration by parts is also applied.

$$\begin{cases} -\left(k(x)\,u'(x)\right)' = f(x) & (\tfrac{1}{5}.1) \\ u(0)=0, \quad u(\ell)=0 & (\tfrac{1}{5}.2) \end{cases}$$

Strong form: Find $u(x)$ in $C_D^2[0,\ell]$

where $C_D^2[0,\ell] = \{ w \in C^2[0,\ell] : w(0)=0, w(\ell)=0 \}$

Weak form derivation:

Multiply (1.1) by $v(x)$ and integrate by parts. We will require $v(0)=0$, $v(\ell)=0$.

Notice that $\left[\left(k(x)\,u'(x)\right)v(x)\right]'$

$$= \left(k(x)\,u'(x)\right)' v(x) + \left(k(x)\,u'(x)\right) v'(x)$$

Then,

$$\int_0^\ell \left(k(x)\,u'(x)\right)' v(x)\,dx = \int_0^\ell \left[\left(k(x)\,u'(x)\right)v(x)\right]' dx - \int_0^\ell k(x)\,u'(x)\,v'(x)\,dx$$

$$= k(x)\,u'(x)\,v(x)\Big|_0^\ell - \int_0^\ell k(x)\,u'(x)\,v'(x)\,dx$$

$$\overset{v(0)=0,\ v(\ell)=0.}{=} 0 - \int_0^\ell k(x)\,u'(x)\,v'(x)\,dx.$$

Then,

$$-\int_0^{\ell} (k(x)u'(x))' v(x)\,dx = \int_0^{\ell} k(x)u'(x)v'(x)\,dx$$

As a result, (1.1) is transformed into

$$\boxed{\int_0^{\ell} k(x)\,u'(x)\,v'(x)\,dx = \int_0^{\ell} f(x)\,v(x)\,dx}.$$

or

$$\boxed{a(u,v) = (f, v)} \qquad (2.1)$$

Weak form of (1.1)

Notice that to satisfy (2.1), we only need to require that $u'(x)$ and $v'(x)$ be integrable in $[0,\ell]$. Good candidates for these functions are the set of piecewise continuous functions in $[0,\ell]$.

So, the original strong formulation can be transformed into the more relax weak formulation:

Find $\vec{u} \in V = \{$ picewise conts. functions in $[0,\ell]\}$

Such that

$$\boxed{a(u,v) = (f, v), \quad \text{for all } v \in V.}$$

Galerkin Method:

Consider $V_n \subset V$ Subspace of V of finite dimension.

Find $v_n \in V_n$ such that (2.1)

$$a(v_n, v) = (f, v), \quad \text{for all } v \in V_n.$$

reduction to a linear system

go to 2_s^1-2_s^2

let $B = \{\phi_1, .., \phi_n\}$ be a basis for V_n

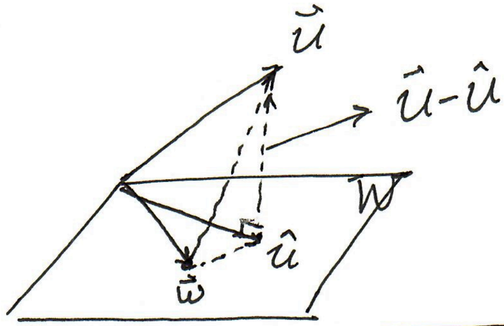
then,
$$v = \sum_{i=1}^{n} c_i \phi_i$$

and (2.1) reduces to

Find $v_n \in V_n$ such that (2.2)

$$a(v_n, \phi_i) = (f, \phi_i), \quad i = 1, 2 \ldots n$$

Best approximation theorem

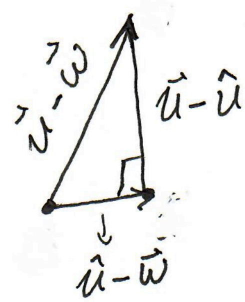


$$\boxed{\|\vec{u}-\hat{u}\| \le \|\vec{u}-\vec{w}\|, \text{ for all } \vec{w}\in W.}$$

Where $\hat{u}$ is the orthogonal projection of $\vec{u}$ in W. definition

This means $(\vec{u}-\hat{u})\cdot\vec{w}=0$, for all $\vec{w}\in W$.

<u>Proof.</u>- The triangle is a right triangle So we can apply Pythagoras theorem because all these vectors belong to a inner-product Space.



$\vec{w}\neq\hat{u}$

Therefore,

$$\|\vec{u}-\vec{w}\|^2 = \|\hat{u}-\vec{w}\|^2 + \|\vec{u}-\hat{u}\|^2$$

Since $\vec{w}\neq\hat{u}$, then $\|\hat{u}-\vec{w}\|>0$

$\Rightarrow \quad \|\vec{u}-\vec{w}\|^2 > \|\vec{u}-\hat{u}\|^2$ or $\|\vec{u}-\vec{w}\| > \|\vec{u}-\hat{u}\|$ for all $\vec{w}\neq\hat{u}$. in W.

$\Rightarrow$ $\hat{u}$ is the best approx. to $\vec{u}$ in W.

If $\quad v = \sum_{i=1}^{n} c_i \phi_i$

then, $a(v_n, v) = \int_0^l k(x)\, v_n'(x)\, v'(x)\, dx$

$$= \int_0^l k(x)\, v_n'(x) \sum_{i=1}^{n} c_i \phi_i'(x)\, dx$$

$$= \sum_{i=1}^{n} c_i \int_0^l k(x)\, v_n'(x)\, \phi_i'(x)\, dx$$

and $\quad (f, v) = \int_0^l f(x) \sum_{i=1}^{n} c_i \phi_i(x)\, dx$

$$= \sum_{i=1}^{n} c_i \int_0^l f(x)\, \phi_i(x)\, dx$$

As a consequence,

$a(v_n, v) = (f, v)$ is equivalent

$$\sum_{i=1}^{n} c_i \Big[\underbrace{\int_0^l k(x)\, v_n'(x)\, \phi_i'(x)\, dx}_{a(v_n, \phi_i)} - \underbrace{\int_0^l f(x)\, \phi_i(x)\, dx}_{(f, \phi_i)} \Big] = 0 \qquad (25.1)$$

and (2.1) reduces to find

Find $v_n \in V_n$ such that

$$\boxed{a(v_n, \phi_i) = (f, \phi_i), \quad i = 1, 2, \ldots n.}$$

In fact,

$$a(v_n, v) = (f, v), \quad \text{for all } v \in V_n \qquad (SS.1)$$

is equivalent to

$$a(v_n, \phi_i) = (f, \phi_i), \quad i = 1, .., n. \qquad (SS.2)$$

The reason is that

if $\quad v = \sum_{i=1}^{n} c_i \phi_i$

then,

(SS.1) Can be written as

$$\sum_{i=1}^{n} c_i \left[\int_0^l k(x)\, v_n'(x)\, \phi_i'(x)\, dx - \int_0^l f(x) \phi_i(x)\, dx \right] = 0 \qquad (SS.3)$$

Since (SS.1) is true for all $v \in V_n$

(SS.3) is true for any Combination of constants C_i's.

In particular,

$C_1 = 1, \; C_i = 0, \; i = 2, \ldots$

$C_2 = 1, \; C_i = 0, \; i \neq 2, \; \ldots, \; C_n = 1, \; C_i = 0, \; i \neq n.$

Then, (SS.3) is equivalent

$$\int_0^l k(x)\, v_n'(x)\, \phi_i'(x)\, dx = \int_0^l f(x)\, \phi_i(x)\, dx$$

or $\boxed{a(v_n, \phi_i) = (f, \phi_i), \text{ for all } i = 1, .., n.}$

Also, $$v_n = \sum_{j=1}^{n} \overset{?}{u_j} \phi_j$$

Subst. into (2.2)

$$a\left(\sum_{j=1}^{n} u_j \phi_j, \phi_i\right) = (f, \phi_i), \quad i=1,..,n$$

or using linearity $$\sum_{j=1}^{n} a(\phi_j, \phi_i)\, u_j = (f, \phi_i), \quad i=1,..,n$$

or $$\boxed{K \hat{u} = \vec{f}} \qquad (3.1)$$

More precisely,

$$\begin{bmatrix} a(\phi_1,\phi_1) & a(\phi_2,\phi_1) & \cdots & a(\phi_n,\phi_1) \\ a(\phi_1,\phi_2) & a(\phi_2,\phi_2) & \cdots & a(\phi_n,\phi_2) \\ \vdots & & & \\ a(\phi_1,\phi_n) & a(\phi_2,\phi_n) & \cdots & a(\phi_n,\phi_n) \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} (f,\phi_1) \\ (f,\phi_2) \\ \vdots \\ (f,\phi_n) \end{bmatrix} \qquad (3.2)$$

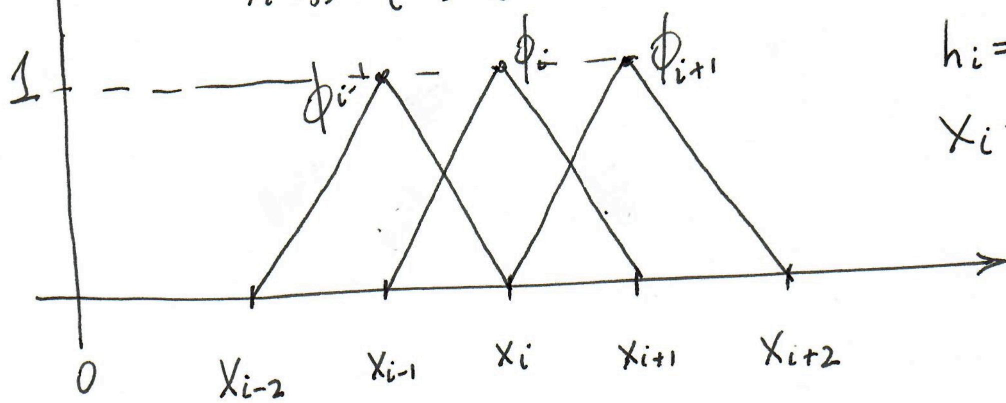
Where $\phi_i(0)=0$ and $\phi_i(\ell)=0$, $i=1,..,n$.
for BVP (1.1)-(1.2) (DIRICHLET/HOMOGENEOUS).
Requirement for
$v(0)=0$, $v(\ell)=0$.

Piecewise Linear Finite Elements.

Consider

$\phi_i(x)$ piecewise linear function satisfying

$\phi_i(x_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$, for a grid $x_0, x_1, .., x_{n-1}$



$h_i = x_i - x_{i-1}$

$x_i = ih, \; i = 0, .., N$

<u>nodes</u>

$\phi_i(x)$ consists of two linear segments. In fact,

$$\boxed{\phi_i(x) = \begin{cases} \phi_i^l(x), & x_{i-1} \leq x \leq x_i \\ \phi_i^r(x), & x_i \leq x \leq x_{i+1} \\ 0, & \text{otherwise.} \end{cases}}$$

where ϕ_i^l is the line thru $(x_{i-1}, 0)$ and $(x_i, 1)$

$$\text{Slope} = \frac{1-0}{x_i - x_{i-1}} = \frac{1}{h} \Rightarrow \boxed{\phi_i^l(x) = \frac{1}{h}(x - x_{i-1})}$$

and ϕ_i^r is the line thru $(x_i, 1)$, and $(x_{i+1}, 0)$

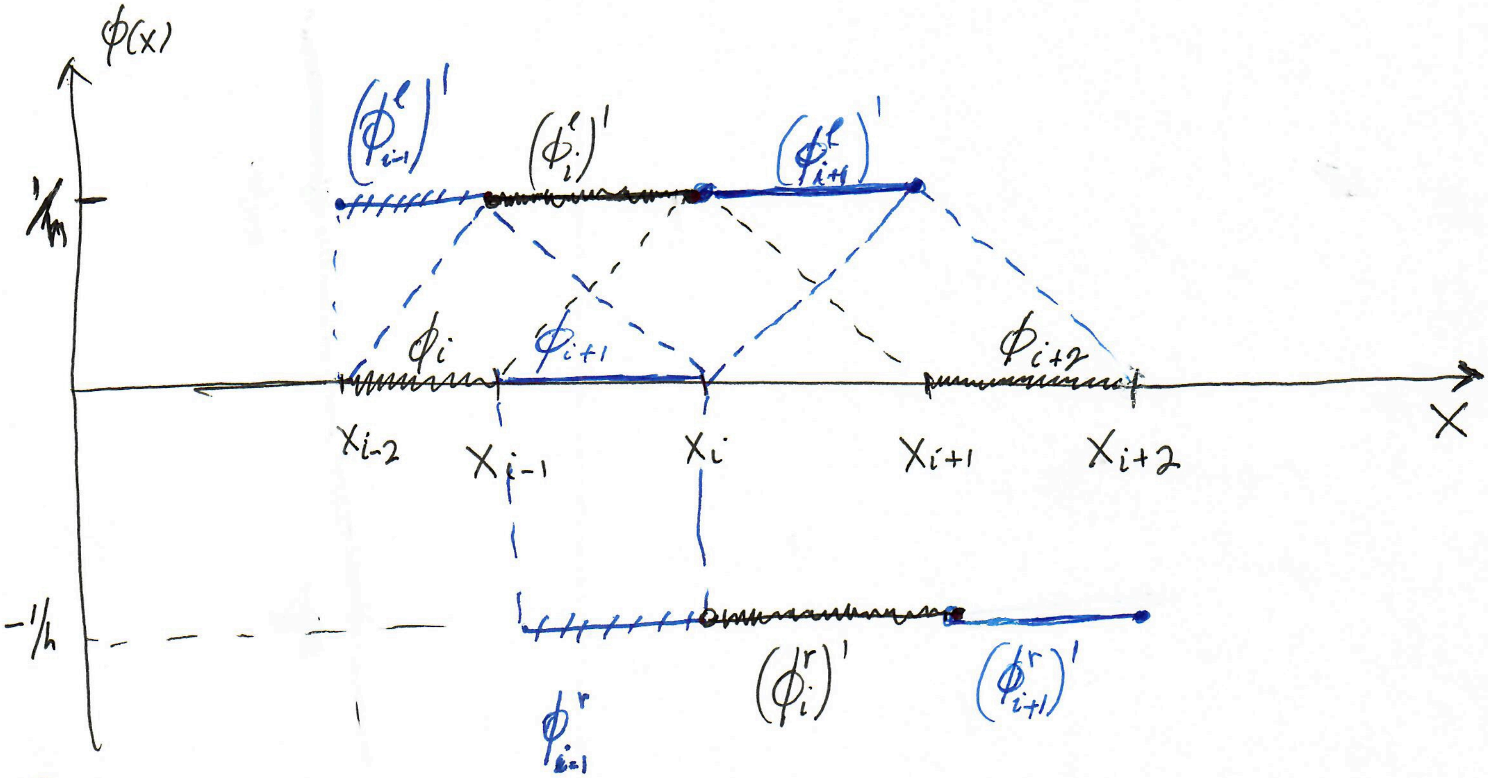
$$\text{Slope} = \frac{-1}{x_{i+1} - x_i} = -\frac{1}{h} \Rightarrow \boxed{\phi_i^r(x) = -\frac{1}{h}(x - x_{i+1})}$$

So,

$$\phi_i(x) = \begin{cases} \frac{1}{h}\left(x-(i-1)h\right), & x_{i-1} \le x \le x_i \\ -\frac{1}{h}\left(x-(i+1)h\right), & x_i \le x \le x_{i+1} \\ 0, & \text{otherwise} \end{cases} \tag{5.1}$$

and

$$\frac{d\phi_i}{dx}(x) = \begin{cases} \frac{1}{h}, & x_{i-1} \le x \le x_i \\ -\frac{1}{h}, & x_i < x \le x_{i+1} \\ 0, & \text{otherwise.} \end{cases}$$



It can be easily proved that the set

$B = \{ \phi_1, \ldots, \phi_{n-1} \}$ is a basis for the vector space of piecewise linear continuous functions S_n given by

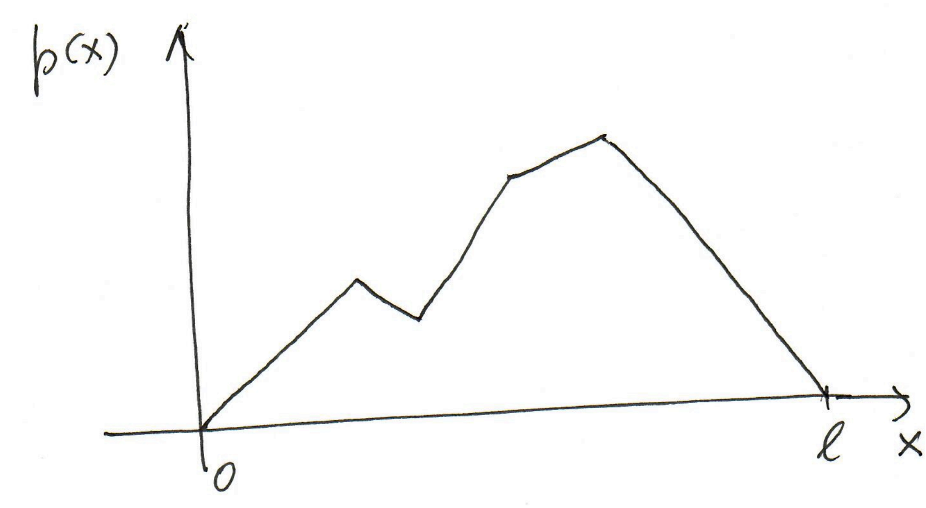
$$S_n = \left\{ p : [0,\ell] \to \mathbb{R} : p \text{ is continuous and piecewise linear, } p(0) = p(\ell) = 0 \text{ relative to a grid consisting of } x_0, x_1, \ldots, x_n \right\}$$

Then, for every $p \in S_n$

Due to ϕ_i definition: $\phi_i(x_i) = 1$.

$$p(x) = \sum_{i=1}^{n-1} c_i \phi_i(x) = \sum_{i=1}^{n-1} p(x_i) \phi_i(x)$$

An example of a function in S_n



Therefore, if we use piecewise linear functions $\phi_i(x)$ as defined by (5.1) for the Galerkin approximation to the Solution of a BVP, the Stiffness matrix of the Galerkin approx. (3.2) reduces to

$$K = \frac{1}{h}\begin{array}{l} i=1 \\ \\ \\ \\ i=n-1 \end{array}\begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & \cdots & -1 & 2 \end{bmatrix} \qquad (6.1)$$

Tridiagonal matrix.

Summary

Strong form:

$$\begin{cases} \left(k(x)\, u'(x) \right)' = f(x), & 0 < x < \ell. \quad (F.1) \\ u(0) = 0, \quad u(\ell) = 0. & (F.2) \end{cases}$$

Find $u \in C_D^2[0,\ell] = \{ w \in C^2[0,\ell] : w(0) = 0, w(\ell) = 0 \}$

Weak form:

$\int_0^\ell (F.1)\, v \, dx \qquad v \in V = \{ v \text{ piecewise conts: } v(0) = 0 = v(\ell) \}$

$$\boxed{a(u,v) = (f,v), \text{ for all } v \in V.}$$

More general $v \in H^1[0,\ell]$. square integrable

Galerkin Method:

$$\boxed{a(v_n, v) = (f, v), \text{ for all } v \in V_n \subset V}$$

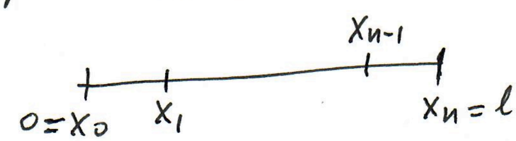
$\hookrightarrow$ finite-dim. (F.3)

v_n is best approximation to u in V. (in V_n)

$a(u - v_n, v) = 0 \qquad v_n$ projection de u in V_n.

Energy norm: in V. $\qquad \|v\| = \left[\int_0^\ell k(x) \left(v'(x) \right)^2 dx \right]^{1/2}.$

The Subspace V_n is defined by creating a mesh or grid



and then defining Same degree polynomials between the grid points, requiring continuity at those grid points. It results piecewise polynomials in $[0,\ell]$.

Dense Linear System of (F.3)

$B = \{\phi_1, \ldots, \phi_n\}$ is a basis of V_n.

Then,
$$v = \sum_{i=1}^{n} c_i \phi_i$$

and (F.3) is equivalent to

$$\boxed{\begin{array}{l}\text{Find } v_n \in V_n \text{ such that} \\ \quad a(v_n, \phi_i) = (f, \phi_i), \quad i = 1, 2, \ldots, n.\end{array}} \qquad (F.4)$$

by expressing $v_n = \sum_{j=1}^{n} u_j \phi_j$

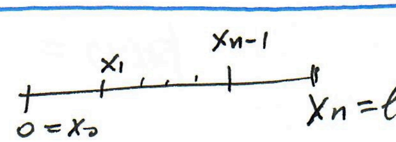
(F.4) is equivalent to

Find the vector $\vec{u} \in \mathbb{R}^n$, such that

$$a\left(\sum_{j=1}^{n} u_j \phi_j, \phi_i\right) = (f, \phi_i), \quad i = 1, \ldots, n.$$

or
$$\boxed{\sum_{j=1}^{n} a(\phi_j, \phi_i) u_j = (f, \phi_i), \quad i = 1, \ldots, n}$$

Given a grid of $n+1$ points



a typical subspace V_n is

$\left\{ \begin{array}{l}\text{Continuous piecewise linear functions in } [0, \ell] \\ \text{relative to the given grid with } p(0) = 0,\ p(\ell) = 0\end{array} \right\}$

Turn

Example:

Strong form:

$$\begin{cases} -\left(K(x)\,u'(x)\right)' = f(x) \\ u(0)=0, \quad u(\ell)=0 \end{cases}$$

Galerkin $\longrightarrow$ bilinear form.

$$a(v_n, v) = (f, v), \quad \text{for all } v \in V_n$$

where $a(v_n,v) = \int_0^\ell k(x)\, v_n'(x)\, v'(x)\, dx$

$$(f,v) = \int_0^\ell f(x)\, v(x)\, dx.$$

Basis $\quad B = \{\phi_1, .., \phi_n\}$ of V_n.

$$v = \sum_{i=1}^{n} c_i \phi_i, \quad v_n = \sum_{j=1}^{n} u_j \phi_j$$

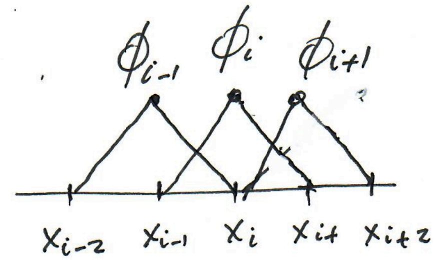
Linear system:

$$\boxed{\sum_{i=1}^{n} a(\phi_j, \phi_i)\, u_j = (f, \phi_i), \quad i = 1, 2, .., n}$$

where $a(\phi_j, \phi_i) = \int_0^\ell k(x)\, \phi_j'(x)\, \phi_i'(x)\, dx$

$$(f, \phi_i) = \int_0^\ell f(x)\, \phi_i(x)\, dx.$$

$$\begin{cases} u''(x) = e^x, & 0 < x < 1 \\ u(0) = 0, \quad u(1) = 0 \end{cases}$$



$$\phi_i(x) = \begin{cases} \frac{1}{h}(x - x_{i-1}), & x_{i-1} \le x \le x_i \\ -\frac{1}{h}(x - x_{i+1}), & x_i \le x \le x_{i+1} \\ 0, & \text{otherwise.} \end{cases} \qquad i = 1,..,n$$

Entries of linear system matrix:

$$a(\phi_j, \phi_i) = \int_0^1 \phi_j'(x)\, \phi_i'(x)\, dx$$

Only nonzero when $j = i-1, i, i+1$

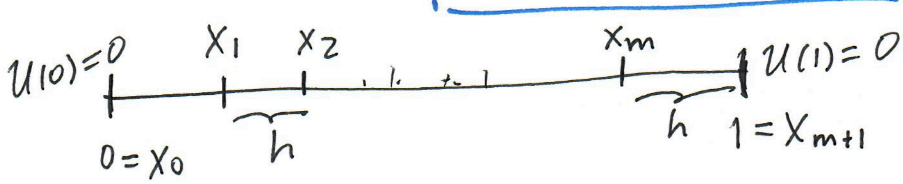
$$a(\phi_{i-1}, \phi_i) = \int_{x_{i-1}}^{x_i} \left(-\frac{1}{h}\right)\left(\frac{1}{h}\right) dx = -\frac{1}{h^2}\underbrace{(x_i - x_{i-1})}_{=h} = -\frac{1}{h}.$$

$$\boxed{a(\phi_{i-1}, \phi_i) = -1/h}$$

$$a(\phi_i, \phi_i) = \int_{x_{i-1}}^{x_{i+1}} (\phi_i')^2(x)\, dx = \frac{1}{h^2}(2h) = \frac{2}{h}$$

$$\boxed{a(\phi_i, \phi_i) = 2/h}$$

$$\boxed{a(\phi_i, \phi_{i+1}) = -1/h}$$



$x_i = ih$

$i = 1,..,m$

Forcing term:

$$(f,\phi_i) = \int_0^1 e^x \phi_i(x)\,dx = \int_{x_{i-1}}^{x_{i+1}} e^x \phi_i(x)\,dx$$

$$= \int_{x_{i-1}}^{x_i} e^x \frac{1}{h}(x-x_{i-1})\,dx + \int_{x_i}^{x_{i+1}} e^x \left(\frac{-1}{h}\right)(x-x_{i+1})\,dx$$

$$\overset{\text{I.P.P.}}{=} e^{x_i}(e^h+e^{-h}-2) = e^{ih}(e^h+e^{-h}-2)$$

$$= d(h)\,e^{ih} \qquad i=1,2..,n.$$

Linear System

$i=1$: $a(\phi_1,\phi_1)u_1 + a(\phi_2,\phi_1)u_2 + 0 + \dots 0 = (f,\phi_1)$

$i=2$: $a(\phi_1,\phi_2)u_1 + a(\phi_2,\phi_2)u_2 + a(\phi_3,\phi_2) + \dots 0 = (f,\phi_2)$

$\vdots$

$3 \leq i < m$: $a(\phi_{i-1},\phi_i)u_{i-1} + a(\phi_i,\phi_i)u_i + a(\phi_{i+1},\phi_i) = (f,\phi_i)$

$\vdots$

$i=m$: $a(\phi_{m-1},\phi_m)u_{m-1} + a(\phi_m,\phi_m)u_m = (f,\phi_m)$

$$\frac{1}{h}\begin{bmatrix} 2 & -1 & 0 & 0 & 0 & \cdots \\ -1 & 2 & -1 & 0 & 0 & \cdots \\ 0 & -1 & 2 & -1 & 0 & \cdots \\ \vdots & & \ddots & \ddots & \ddots & \\ 0 & 0 & \cdots & & -1 & 2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ \vdots \\ u_m \end{bmatrix} = \frac{d(h)}{h}\begin{bmatrix} e^{h} \\ e^{2h} \\ \vdots \\ e^{mh} \end{bmatrix}$$

<u>Exact Soln:</u> $u'(x) = -e^{x} + c_1, \quad u(x) = -e^{x} + c_1 x + c_2$

$0 = u(0) = -1 + c_2 \Rightarrow \boxed{c_2 = +1} \Rightarrow u(x) = -e^{x} + c_1 x + 1$

$0 = u(1) = -e + c_1 + 1 \Rightarrow \boxed{c_1 = -1 + e}$

$$\Rightarrow \boxed{u(x) = -e^{x} - (1-e)\,x + 1}$$