Chapter 1: Finite Difference Approximation

Consider the IBVP: (Heat conduction).

$$\begin{cases} v_t + k v_{xx} = f(x,t), & 0 < x < L, \; t > 0. \\ v(0,t) = A, \quad v(L,t) = B \\ v(x,0) = g(x) \end{cases}$$

Domain:

If we assume $t \to \infty$ and $f(x,t) = f(x)$ (indep. of t)
All the initial effects disappear and we are lead to the one-dimensional two point BVP:

Equillibrium BVP or Steady state BVP.

$$\begin{cases} v''(x) = F(x), & 0 < x < L \quad (1.1) \\ v(0) = A, \quad v(L) = B & (1.2) \end{cases}$$

where $F(x) = -f(x)/k$

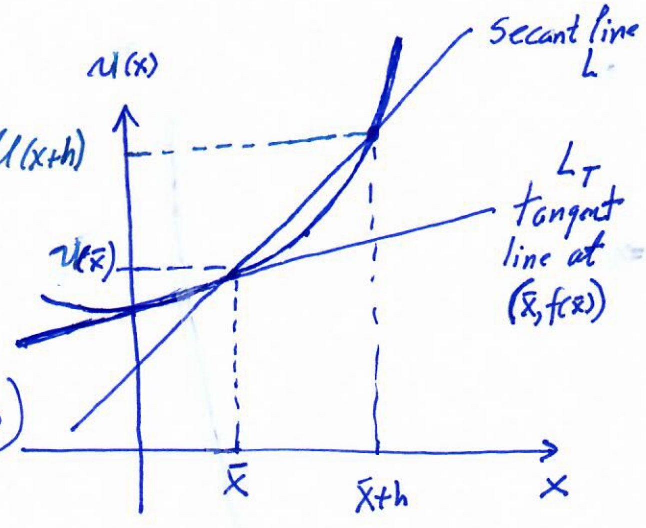
Remark: Before attempting any method to obtain a solution, we should analize if ① the BVP has a solution and ② if this solution is unique.

This will be discussed later.

The BVP (1) can be treated analytically or numerically.

A numerical approach called "finite-difference" consists of approximating all derivatives present in (1.1) by discrete formulas based on values of v at neighbor points. An important aspect of the approximation is the analysis of how good the approximation is. This leads to the concept of <u>order of accuracy</u>

<u>Approximation of $u'(\bar{x})$:</u>



We know from calculus that

$u'(\bar{x}) =$ Slope of L_T $(h \to 0)$

On the other hand,

$$\text{Slope of } L = \frac{u(\bar{x}+h) - u(\bar{x})}{h}$$

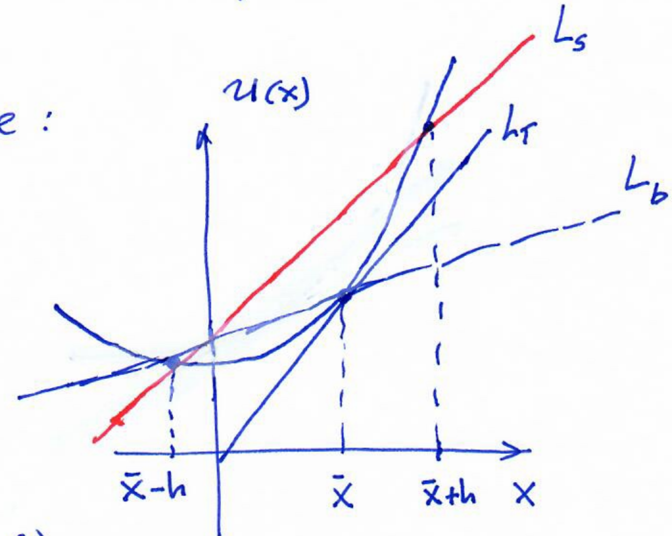
and by definition

$$u'(\bar{x}) = \lim_{h \to 0} \frac{u(\bar{x}+h) - u(\bar{x})}{h}$$

So a "good" approximation for $u'(\bar{x})$ might be

$$u'(\bar{x}) \simeq \frac{u(\bar{x}+h) - u(\bar{x})}{h} \quad \text{for small } h.$$

Two other possibilities are:



$$u'(\bar{x}) \simeq \frac{u(\bar{x}) - u(\bar{x}-h)}{h}$$

= slope of L_b

and

$$u'(x) \simeq \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h} \; 2 = \text{slope of } L_s.$$

<u>Definition</u>

$$D_+ u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x})}{h}, \quad \text{forward approximation}$$

$$D_- u(\bar{x}) = \frac{u(\bar{x}-h) - u(\bar{x})}{h}, \quad \text{backward} \quad "$$

$$D_0 u(\bar{x}) = \frac{u(\bar{x}+h) - u(\bar{x}-h)}{2h}, \quad \text{Centered approximation.}$$

From the previous graphs, we conclude that Centered approximation should be better than forward and backward. Also, that forward and backward should be similar.

We also consider another approximation of $u'(\bar{x})$

$$D_3 u(\bar{x}) = \frac{1}{6h}\left[2u(\bar{x}+h) + 3u(\bar{x}) - 6u(\bar{x}-h) + u(\bar{x}-2h)\right]$$

Two important questions are:

a) How "good" are these approximations?
Can we measure it?

b) Is there a consistent method to obtain them?

<u>Answers:</u> How "good"?

Consider the <u>example 1.1</u>.

For $u(x) = \sin x$, $\bar{x} = 1$

we would like to approximate $u'(\bar{x}) = u'(1) = \cos(1)$
$= 0.5403023..$

using
$D_+u(1)$, $D_-u(1)$, $D_0u(1)$, and $D_3u(1)$.

Run experiment MATLAB.

Table 1.1

$f(x) =$

$f'(x) = \cos x$

at $x = 1$

h	D_+	D_0	D_3
10^{-1}	0.42×10^{-1}	0.09×10^{-2}	0.07×10^{-3}
10^{-2}	0.42×10^{-2}	0.09×10^{-4}	0.07×10^{-6}
10^{-3}	0.42×10^{-3}	0.09×10^{-6}	0.07×10^{-9}
	$0.42 h$	$0.09 h^2$	$0.07 h^3$

Table 1.1. $f(x) = e^x$, $f'(x) = e^x$, $x = 5$.

h	D_+	D_0	D_3
10^{-1}	$\simeq 74 \times 10^{-1}$	24.7×10^{-2}	12×10^{-3}
10^{-2}	74×10^{-2}	24.7×10^{-4}	12×10^{-6}
10^{-3}	74×10^{-3}	24.7×10^{-6}	12×10^{-9}
	$74 h$	$24.7 h^2$	$12 h^3$

Try $f(x) = e^x$ at $x = 10$
and refine $\underline{h}$ values.

$E(h) =$ Error term for $D_3 u$ $= \dfrac{u^{(iv)}(x)}{12} h^3$

for e^x, $x = 5$: $E(h)\ \dfrac{e^5}{12} h^3 \simeq 12.4 h^3$

From Table 1, we conclude

h	$D_+u(1) - \underset{\simeq u'(1)}{\cos(1)}$	$D_0u(1) - \underset{\simeq u'(1)}{\cos(1)}$	$D_3u(1) - \underset{\simeq u'(1)}{\cos(1)}$
10^{-1}	$\simeq -0.4 \times 10^{-1}$	$\simeq -0.09 \times (10^{-1})^2$	$\simeq 0.07 \times (10^{-1})^3$
10^{-2}	$\simeq -0.4 \times 10^{-2}$	$\simeq -0.09 \times (10^{-2})^2$	$\simeq 0.07 \times (10^{-2})^3$
10^{-3}	$\simeq -0.4 \times 10^{-3}$	$\simeq -0.09 \times (10^{-3})^2$	$\simeq 0.07 \times (10^{-3})^3$

or

$$D_+u(\bar{x}) - u'(\bar{x}) \simeq -0.42\,h = E(h) \quad \text{error for stepsize } h.$$

$$D_0u(\bar{x}) - u'(\bar{x}) \simeq -0.09\,h^2 = E(h)$$

$$D_3u(\bar{x}) - u'(\bar{x}) \simeq 0.07\,h^3 = E(h).$$

In all these cases, the error $E(h)$ behaves like

$$E(h) = c\,h^p. \qquad (5.1)$$

In general, we will say that a finite difference approximation to $u'(\bar{x})$ has an <u>order of accuracy</u> equal to p if (5.1) holds.

This analysis is very particular. It can be done more general and also we can obtain analytical expression for the error for any smooth function $u(x)$.

Notice that applying ln to both sides of |(5.1)|

$$\ln(|E(h)|) = \ln(|c|) + p\ln(h).$$

It means if an error function $E(h)$ is graphed using a ln-ln scale and the points lie along a line then the slope of that line is the order of accuracy of the approximation.

In our previous example,

$E(h)$ and h for $\boxed{D_0 u(\bar{x})}$ satisfy

$$\ln(E(h)) = \ln(0.09) + 2\ln(h).$$

Even if we do not know "p" beforehand,
The slope $p = 2$ can be easily obtained considering the two points $(h, E(h))$ $(10^{-1}, -9\times10^{-4})$, $(10^{-2}, -9\times10^{-6}) = (h, E(h))$

$$p = \frac{\log(|-9\times10^{-4}|) - \log(|-9\times10^{-6}|)}{\log(10^{-1}) - \log(10^{-2})} = \frac{\log(9)+\log(10^{-4}) - (\log(9)+\log(10^{-6}))}{-1-(-2)}$$

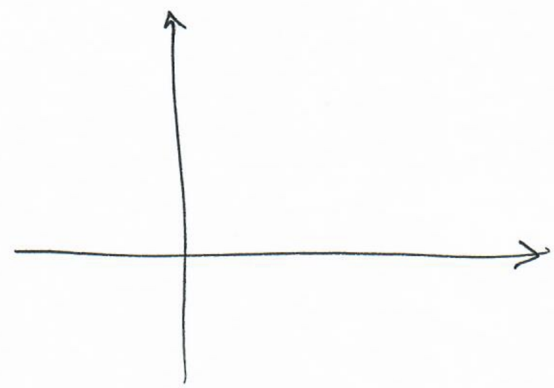
$$= \frac{-4-(-6)}{1} = 2.$$

Example 2.- Apply to $u(x) = e^x$; $x = 1$.

A useful application of this technique to determine if an approximation of a numerical solution to an exact solution is of $\mathcal{O}(h^p)$. $p = ?$.

Show paper's result

N -	h -	E(h)
6	$\frac{2\pi}{60}$	1.12e-1
12	$\frac{2\pi}{120}$	2.75e-2
241	$\frac{2\pi}{240}$	6.91e-3



Slope of line

1) $$p = \frac{\ln(2.75e\text{-}2) - \ln(1.12e\text{-}1)}{\ln\left(\frac{\pi}{60}\right) - \ln\left(\pi/30\right)}$$

$$= \frac{\ln(2.75e\text{-}2) - \ln(1.12e\text{-}1)}{\ln\left(\frac{\pi/60}{\pi/30}\right) = \ln(1/2) = -\ln(2)} \simeq 2.026$$

2) Also, $$\frac{\ln(2.75e\text{-}2) - \ln(6.91e\text{-}3)}{\ln(\pi/60) - \ln(\pi/120)} \simeq 1.99.$$