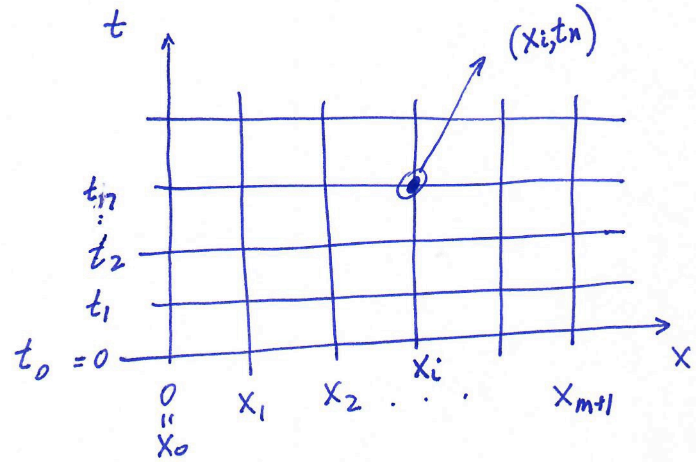
CHAPTER 9 Diffusion Equations and Parabolic Problems.

Consider IBVP:

$$
\begin{cases}
u_t = \sigma u_{xx}, & 0<x<1, \quad \sigma>0. & (1)\\
u(x,0) = \eta(x) & & (2)\\
u(0,t) = g(t), & t>0 & (3)\\
u(1,t) = h(t), & t>0 & (4)
\end{cases}
$$

Discretization : Grid



$x_i = ih$, $i=0,\dots,m+1$

$t_n = nk$, $n=0,\dots,N_{max}$

$h = \Delta x$

$k = \Delta t$

$U_i^n \simeq u(x_i, t_n)$

FT-CS numerical method

(I) Approx. of $(u_t)_i^n$ (forward difference).

$$u(x_i, t_{n+1}) = u_i^{n+1} = u_i^n + k(u_t)_i^n + \frac{k^2}{2}(u_{tt})_i^{n+\theta}$$

$$\Rightarrow \quad (u_t)_i^n = \frac{u_i^{n+1} - u_i^n}{k} - \frac{k}{2}(u_{tt})_i^{n+\theta} \qquad (5)$$

$0<\theta<1$

Similarly, centered approx. of $(u_{xx})_i^n$

$$\text{II} \quad (u_{xx})_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} - \frac{h^2}{12}(u_{xxxx})_{i+\xi}^n \qquad (6)$$

$0<\xi<1$

Subst. (5)-(6) into (1)

$$\frac{u_i^{n+1} - u_i^n}{K} - \frac{K}{2}(u_{tt})_i^{n+\theta} = \sigma\left[\frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{h^2} - \frac{h^2}{12}(u_{xxxx})_{i+\xi}^n\right]$$

$i=1,2,.,m$

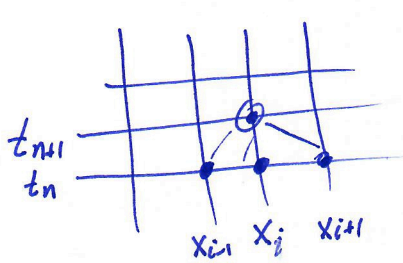
Neglecting discretization errors, it leads to

$$\boxed{\frac{v_i^{n+1} - v_i^n}{K} = \frac{\sigma}{h^2}\left(v_{i+1}^n - 2v_i^n + v_{i-1}^n\right)} \qquad (6.1)$$

Explicit method

$$\boxed{v_i^{n+1} = v_i^n + \frac{\sigma K}{h^2}\left(v_{i+1}^n - 2v_i^n + v_{i-1}^n\right),} \quad i=1,2,\ldots m. \qquad (7)$$

FT–CS finite difference method for Heat Cond. (1-D).



STENCIL

Explain "marching in time" process.

by calling $r \equiv \frac{\sigma k}{h^2}$ (7) can be written as

$$\boxed{v_i^{n+1} = r\, v_{i-1}^n + (1-2r)\, v_i^n + r\, v_{i-1}^n,} \qquad (7.1) \quad i = 1, 2, \ldots, m$$

Interior points — Explicit scheme

Dirichlet BCs: $v_0^n = g(t_n)$, $v_{m+1}^n = h(t_n)$, $n = 1, 2, \ldots, N_{max}$

IC: $v_i^0 = \eta(x_i)$, $i = 1, 2, \ldots, m$

Book Experiment: Example 2.3.1

$$\begin{cases} u_t = \sigma u_{xx}, & 0 < x < 1, \; t > 0 \\ u(0,t) = 0, & u(1,t) = 0, \; t > 0 \\ u(x,0) = \begin{cases} 2x, & 0 \le x \le 1/2 \\ 2(1-x), & 1/2 \le x \le 1 \end{cases} \end{cases}$$

Exact Soln:

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \frac{8 \sin(k\pi/2)}{(k\pi)^2}\, e^{-k^2\pi^2 t} \sin(k\pi x).}$$

Run the following experiments:

Book Exp. 1:

$$\begin{cases} \Delta t = 0.001 \\ \Delta x = 0.1 \\ \sigma = 1 \end{cases} \Rightarrow r = \frac{\sigma \Delta t}{\Delta x^2} = 0.1$$

Book Exp. 2:

$$\begin{cases} \Delta t = 0.01 \\ \Delta x = 0.1 \\ \sigma = 1 \end{cases} \Rightarrow r = \frac{\sigma \Delta t}{\Delta x^2} = 1$$

Other Experiments:

Experiment 3:

$$\begin{matrix} \Delta t = 0.0008 \\ \Delta x = 0.05 \\ \sigma = 1 \end{matrix} \Rightarrow \boxed{r = 0.32}$$

Experiment 4:

$$\begin{matrix} \Delta t = 0.001 \\ \Delta x = 0.05 \\ t_{max} = 0.3 \\ \sigma = 1 \end{matrix} \Rightarrow \boxed{r = 0.4}$$

Experiment 5:

$$\begin{matrix} \Delta t = 0.0012 \\ \Delta x = 0.05 \\ \sigma = 1 \\ t_{max} = 0.3 \end{matrix} \Rightarrow \boxed{r = 0.48}$$

Exp. 6:

$$\begin{matrix} \Delta t = 0.00125 \\ \Delta x = 0.05 \\ \sigma = 1 \\ t_{max} = 0.3 \end{matrix} \Rightarrow \boxed{r = 0.5}$$

Code Exp 3

Exp. 7

$$\begin{matrix} \Delta t = 0.00129 \\ \Delta x = 0.05 \\ \sigma = 1 \end{matrix} \Downarrow \boxed{r = 0.516}$$

Code Exp 4

converge to that of the partial differential equation. Therefore, the domain of dependence of the difference scheme must contain that of the partial differential equation. □

Problems

1. Suppose a is a positive constant. The scheme (2.2.4) only depends on the Courant number α_j^n which, for Examples 2.2.1 and 2.2.2, is the constant $a\Delta t/\Delta x$. Is there a particular choice of the Courant number on $(0, 1]$ that produces more accurate solutions than others? Answer the same question for (2.2.2b) when $a < 0$.

2. What restrictions, if any, should be placed on the Courant number α_j^n for the forward time-centered space scheme (2.2.5) to satisfy the Courant, Friedrichs, Lewy theorem? We must, of course, study the behavior of (2.2.5); however, prior to doing this in Chapter 3, experiment by applying the method to the problems of Examples 2.2.1 and 2.2.2. Use the same mesh spacings as the earlier examples.

2.3 A Simple Difference Scheme for the Heat Equation

Let us determine whether similar or different phenomena occur when solving a simple initial-boundary value problem for the heat conduction equation. In particular, consider

$$u_t = \sigma u_{xx}, \qquad 0 < x < 1, \qquad t > 0, \tag{2.3.1a}$$

$$u(x, 0) = \phi(x), \qquad 0 \le x \le 1, \tag{2.3.1b}$$

$$u(0, t) = u(1, t) = 0, \tag{2.3.1c}$$

where the diffusivity σ is positive.

In order to construct finite-difference approximations of (2.3.1): (i) introduce a uniform grid of spacing $\Delta x = 1/J$, $J > 0$, and Δt on the strip $(0, 1) \times (t > 0)$ (Figure 2.3.1); (ii) evaluate (2.3.1a) at the mesh point $(j\Delta x, n\Delta t)$; and (iii) replace the partial derivatives by forward time (2.1.8) and centered space (2.1.9) differences to obtain

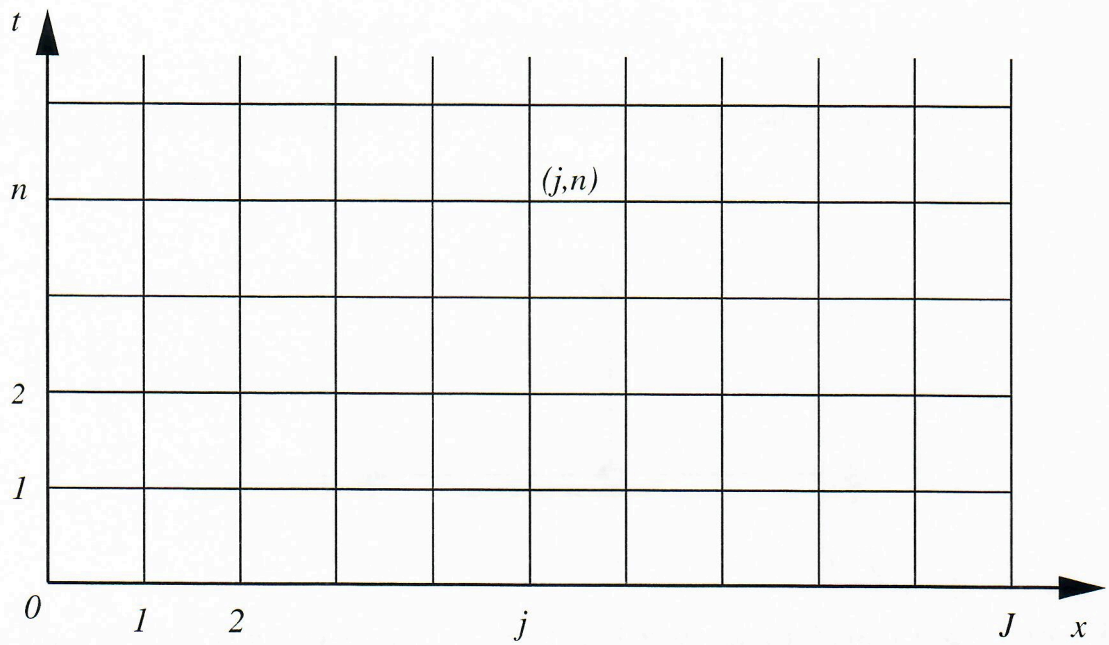


Figure 2.3.1: Computational grid used for the finite difference solution of the heat conduction problem (2.3.1).

$$\frac{u_j^{n+1} - u_j^n}{\Delta t} - \frac{\Delta t}{2}(u_{tt})_j^{n+\theta} = \sigma\left[\frac{u_{j+1}^n - 2u_j^n + u_{j-1}^n}{\Delta x^2} - \frac{\Delta x^2}{12}(u_{xxxx})_{j+\xi}^n\right], \tag{2.3.2a}$$

where $-1 < \xi < 1$, $0 < \theta < 1$. (With second spatial derivatives in (2.3.1) there seems little point in using forward or backward spatial derivatives and we have not done so.)

Neglecting the discretization error terms, we get the finite difference scheme

$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \sigma \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{\Delta x^2} \tag{2.3.2b}$$

or, solving for U_j^{n+1},

$$U_j^{n+1} = rU_{j-1}^n + (1-2r)U_j^n + rU_{j+1}^n, \tag{2.3.3a}$$

where

$$r = \sigma \frac{\Delta t}{\Delta x^2}. \tag{2.3.3b}$$

The initial and boundary conditions (2.3.1b, 2.3.1c) are

$$U_j^0 = \phi(j\Delta x), \qquad j = 0, 1, \ldots, J, \tag{2.3.3c}$$

t	n	$x = 0$ $j = 0$	0.1 1	0.2 2	0.3 3	0.4 4	0.5 5	0.6 6
0.0	0	0.0	0.2	0.4	0.6	0.8	1.0	0.8
0.001	1	0.0	0.2	0.4	0.6	0.8	0.96	0.8
0.002	2	0.0	0.2	0.4	0.6	0.796	0.928	0.796
0.003	3	0.0	0.2	0.4	0.600	0.790	0.901	0.790
0.004	4	0.0	0.2	0.4	0.599	0.782	0.879	0.782
0.005	5	0.0	0.2	0.400	0.597	0.773	0.860	0.773
0.006	6	0.0	0.2	0.400	0.595	0.764	0.842	0.764
0.007	7	0.0	0.200	0.399	0.592	0.755	0.827	0.755
0.008	8	0.0	0.200	0.399	0.589	0.746	0.813	0.746
0.009	9	0.0	0.200	0.398	0.586	0.737	0.799	0.737
0.010	10	0.0	0.200	0.397	0.582	0.728	0.787	0.728

Table 2.3.1: Solution of Example 2.3.1 using the forward time-centered space scheme (2.3.3a) with $r = 0.1$.

t	n	$x = 0$ $j = 0$	0.1 1	0.2 2	0.3 3	0.4 4	0.5 5	0.6 6
0.0	0	0.0	0.2	0.4	0.6	0.8	1.0	0.8
0.01	1	0.0	0.2	0.4	0.6	0.8	0.6	0.8
0.02	2	0.0	0.2	0.4	0.6	0.4	1.0	0.4
0.03	3	0.0	0.2	0.4	0.2	1.2	-0.2	1.2
0.04	4	0.0	0.2	0.0	1.4	-1.2	2.6	-1.2

Table 2.3.2: Solution of Example 2.3.1 using the forward time-centered space scheme (2.3.3a) with $r = 1$.

t	n	$x = 0.3 (j = 3)$ $\lvert u_3^n - U_3^n \rvert$	$x = 0.5 (j = 5)$ $\lvert u_5^n - U_5^n \rvert$
0.005	5	0.0008	0.023
0.01	10	0.004	0.016
0.1	100	0.011	0.012

do the same for r=1.

Table 2.3.3: Errors at $x = 0.3$ ($j = 3$) and $x = 0.5$ ($j = 5$) for the solution of Example 2.3.1 using the forward time-centered space scheme (2.3.3a) with $r = 0.1$.

shown in Tables 2.3.1 and 2.3.2 for a few time steps. Errors of the solution with $r = 0.1$ at $x = 0.3$ and 0.5 are presented in Table 2.3.3 for a few times. The solutions are also shown in Figure 2.3.3.

As shown in Tables 2.3.1 and 2.3.3, the solution of (2.3.3a) with $r = 0.1$ is producing a reasonable approximation of the exact solution. The larger errors at $x = 0.5$ for

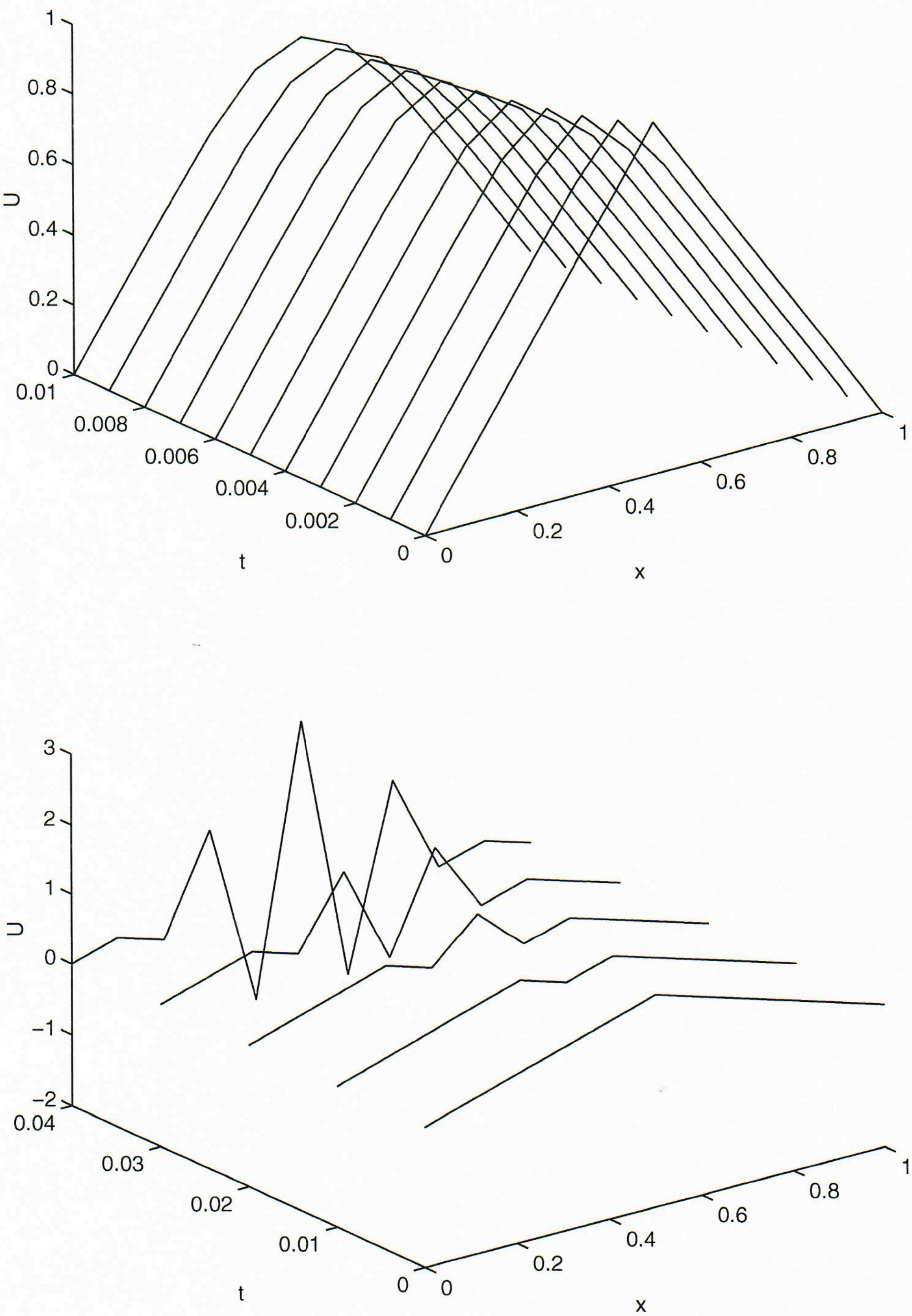
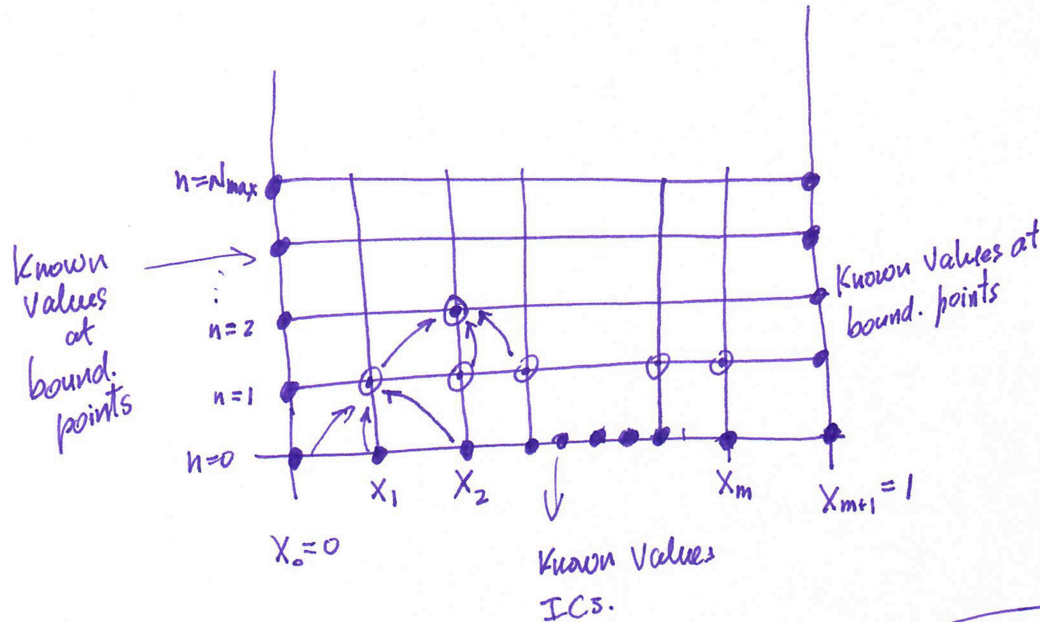


Figure 2.3.3: Solutions of Example 2.3.1 obtained by the forward time-centered space scheme (2.3.3a) with $r = 0.1$ (top) and $r = 1$ (bottom).



Jump to experiments in ②'.

Local truncation error and Consistency

Let's define

Continuous differential operator:
$$\boxed{P v \equiv v_t - \sigma v_{xx}} \qquad (8)$$

Discrete finite difference operator:
$$\boxed{P_\Delta v_i^n \equiv \frac{v_i^{n+1} - v_i^n}{k} - \sigma \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h^2}} \qquad (9)$$

By substituting a sufficiently smooth function $v(x,t)$ into (9)

$$\boxed{P_\Delta v_i^n = \frac{v_i^{n+1} - v_i^n}{k} - \sigma \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h^2}} \qquad (10)$$

Now,

$$\frac{v_i^{n+1} - v_i^n}{k} = (v_t)_i^n + \frac{k}{2}(v_{tt})_i^{n+\theta}, \qquad 0<\theta<1 \quad (11)$$

$$\frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{h^2} = (v_{xx})_i^n + \frac{h^2}{12}(v_{xxxx})_{i+\xi}^n, \quad 0<\xi<1 \quad (12)$$

Therefore, substituting (11) and (12) into (10)

$$P_\Delta v_i^n = (v_t)_i^n + \frac{k}{2}(v_{tt})_i^{n+\theta} - \sigma (v_{xx})_i^n - \sigma\frac{h^2}{12}(v_{xxxx})_{i+\xi}^n$$

or

$$P_\Delta v_i^n = P v(x_i, t_n) + \left[\frac{k}{2}(v_{tt})_i^{n+\theta} - \sigma\frac{h^2}{12}(v_{xxxx})_{i+\xi}^n\right]$$

$$\Rightarrow \boxed{P v(x_i, t_n) - P_\Delta v_i^n = -\frac{k}{2}(v_{tt})_i^{n+\theta} + \frac{\sigma h^2}{12}(v_{xxxx})_{i+\xi}^n}$$

Definition:- The Local discretization error of the FT-CS difference approx. for the heat equation differential operator

is $\boxed{\tau_i^n \equiv P v(x_i, t_n) - P_\Delta v_i^n = -\frac{k}{2}(v_{tt})_i^{n+\theta} + \frac{\sigma h^2}{12}(v_{xxxx})_{i+\xi}^n}$

In general,

if Pv is a continuous differential operator

and $P_\Delta v_i^n$ is a discrete finite difference operator

Definition:- The local discretization error of the finite difference $P_\Delta v_i^n$ approximation for the continuous operator Pv is

$$\boxed{\tau_i^n \equiv P v_i(x_i, t_n) - P_\Delta v_i^n} \qquad (5.0)$$

Differential Equation defined by differential operator "P"

$$\boxed{P u(x,t) = 0} \qquad (5.1)$$

Finite difference scheme defined by discrete finite diff. operator "P_Δ"

$$\boxed{P_\Delta v_i^n = 0} \qquad (5.2)$$

Definition:- (consistency)

A finite difference scheme (5.2) is consistent with a PDE (5.1), if the local discretization error tends to zero as $\Delta x \to 0$ and $\Delta t \to 0$.

Convergence

Definition:

A finite difference approximation $\vec{v}^n = \begin{bmatrix} v_0^n \\ v_1^n \\ \vdots \\ v_J^n \end{bmatrix}$ Converges to the Solution of a partial differential equation $\vec{u}^n = \begin{bmatrix} u_0^n \\ u_1^n \\ \vdots \\ u_J^n \end{bmatrix}$ subject to Initial or boundary conditions.

$$u_j^n = u(x_j, t_n)$$

on a time $0 < t \leq T$ in a particular vector norm if

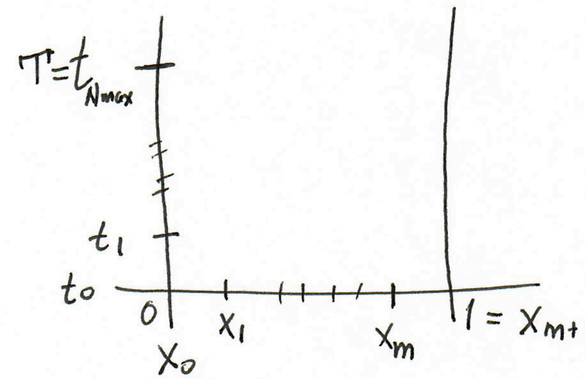
$$\|\vec{u}^n - \vec{v}^n\| \to 0, \quad n \to \infty, \ \Delta x \to 0, \ \Delta t \to 0$$
$$n\Delta t \leq T.$$

Remark: Convergence implies that the discrete and continuous Solutions approach each other for $t \in (0,T)$ in a particular vector norm as the mesh spacing decreases.

Convergence of Forward-time and Centered-space for Heat Equation

$$\begin{cases} u_t = \sigma u_{xx}, & 0 < x < 1,\ t > 0 \\ u(0,t) = g(t), & u(1,t) = h(t) \\ u(x,0) = \phi(x) \end{cases} \qquad (1.1)$$

Discretization:



$$(u_t)_i^n = \frac{u_i^{n+1} - u_i^n}{\Delta t} - \frac{\Delta t}{2}(u_{tt})_i^n + O(\Delta t^2).$$

Forward in time approx. for $(u_t)_i^n$

$$(u_{xx})_i^n = \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2} - \frac{\Delta x^2}{12}(u_{4x})_i^n + O(\Delta x^4).$$

Centered-space approx. for $(u_{xx})_i^n$

Then,

$$\frac{u_i^{n+1} - u_i^n}{\Delta t} - \sigma \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}$$

$$= \left[(u_t)_i^n - \sigma (u_{xx})_i^n\right] + \underbrace{\boxed{\frac{\Delta t}{2}(u_{tt})_i^n - \sigma \frac{\Delta x^2}{12}(u_{4x})_i^n + O(\Delta t^2) + O(\Delta x^4)}} \qquad (1.2)$$

$$\parallel$$

LTE = Local Truncation error $= T_i^n$

Eliminating L.T.E., we arrive to the finite difference scheme for the Heat Equ. given by

$$\frac{v_i^{n+1} - v_i^n}{\Delta t} = \sigma \frac{v_{i+1}^n - 2v_i^n + v_{i-1}^n}{(\Delta x)^2}, \quad \begin{matrix} i = 1, .., m \\ n = 1, .., N_{max} \end{matrix}$$

or

$$\boxed{v_i^{n+1} = r v_{i+1}^n + (1-2r) v_i^n + r v_{i-1}^n} \qquad (2.1)$$

where $\quad r \equiv \frac{\sigma \Delta t}{(\Delta x)^2}$ $\qquad$ $\boxed{\text{FT-CS}}$

The discrete system of equations is completed with

$$v_0^n = g(t_n), \quad v_{m+1}^n = h(t_n), \quad n = 0, .., N_{max}$$

$$v_i^0 = \phi(x_i), \quad i = 1, .., m$$

Also, if $u(x,t)$ is a solution of heat equation (1.2) can be written as

$$\boxed{u_i^{n+1} = r u_{i+1}^n + (1-2r) u_i^n + r u_{i-1}^n - \Delta t \, \tau_i^n} \qquad (2.2).$$

Defining $\boxed{e_i^n \equiv u_i^n - v_i^n,}$ $\vec{e}^n = \begin{bmatrix} e_0^n \\ e_1^n \\ \vdots \\ e_{m+1}^n \end{bmatrix}$

for convergence, we need

$$\|\vec{e}^n\| \longrightarrow 0$$

$\Delta x \to 0,\ \Delta t \to 0$

$n \to \infty,\ n\Delta t \le T.$

<u>Thm</u> The numerical scheme FT-CS for the heat equation IBVP (1.1) Converges in $\|\cdot\|_\infty$ if $r \le 1/2$.

<u>Proof.</u> Subtracting (2.1) from (2.2).

$$e_i^{n+1} = u_i^{n+1} - v_i^{n+1} = r\, e_{i+1}^n + (1-2r)\, e_i^n + r\, e_{i-1}^n - \Delta t\, \tau_i^n$$

$i = 1, 2, .., m$

Then applying triang inequality

$$|e_i^{n+1}| \le r\,|e_{i+1}^n| + |1-2r|\,|e_i^n| + r\,|e_{i-1}^n| + \Delta t\,|\tau_i^n| \quad (*)$$

if $\|e^n\|_\infty = \max_i |e_i^n|$, $\|\tau^n\|_\infty = \max_i |\tau_i^n|$

then, $|e_i^{n+1}| \le \left(r + |1-2r| + r\right)\|\vec{e}^{\,n}\|_\infty + \Delta t\,\|\vec{\tau}^{\,n}\|_\infty \quad i=1,2,\ldots,m$ (*)

If $r \le 1/2 \Rightarrow 1-2r \geqslant 0$ and $|1-2r| = 1-2r$

$$|e_i^{n+1}| \le \|\vec{e}^{\,n}\|_\infty + \Delta t\,\|\vec{\tau}^{\,n}\|, \quad i=1,2,\ldots,m$$

$$\Rightarrow \boxed{\|\vec{e}^{\,n+1}\|_\infty \le \|\vec{e}^{\,n}\|_\infty + \Delta t\,\|\vec{\tau}^{\,n}\|_\infty}, \qquad (**)$$

Iterating $\|\vec{e}^{\,n+1}\|_\infty \le \|\vec{e}^{\,n}\|_\infty + \Delta t\,\|\vec{\tau}^{\,n}\|_\infty \le \|\vec{e}^{\,n-1}\|_\infty + \Delta t\left(\|\vec{\tau}^{\,n}\|_\infty + \|\vec{\tau}^{\,n-1}\|_\infty\right)$

$$\ldots\; \|\vec{e}^{\,n+1}\|_\infty \le \|\vec{e}^{\,0}\|_\infty + \Delta t\left(\|\vec{\tau}^{\,n}\|_\infty + \cdots + \|\vec{\tau}^{\,0}\|_\infty\right) \le \|\vec{e}^{\,0}\|_\infty + (n+1)\tau\,\Delta t$$

where $\tau \overset{def}{=} \max_{0\le k\le n} \|\vec{\tau}^{\,k}\|_\infty$ $\qquad \|\vec{e}^{\,n+1}\|_\infty \le \|\vec{e}^{\,0}\|_\infty + (n+1)\tau\,\Delta t$

In particular, $\|\vec{e}^{\,n}\|_\infty \le n\tau\Delta t \le \tau T$

Since $\|\vec{e}^{\,0}\| \xrightarrow{ICs.} 0$ and $n\Delta t \le T \Rightarrow$

$$\Rightarrow \|\vec{e}^{\,n}\|_\infty \le T\tau.$$

$$\boxed{r \equiv \frac{\sigma\Delta t}{\Delta x^2} \le 1/2 \;\Rightarrow\; \boxed{\Delta t \le \frac{1}{2\sigma}(\Delta x)^2}}$$

Severe restriction in time step, since $(\Delta x)^2$ may be very small.

Now $\tau \le \frac{\Delta t}{2} K + \sigma \frac{\Delta x^2}{12} M$ ✓ (Using definition of truncation error (1.2))

triang. inequality.

$$K = \max_{\substack{0 \le t \le T \\ 0 \le x \le 1}} |u_{tt}|, \qquad M = \max_{\substack{0 \le t \le T \\ 0 \le x \le 1}} |u_{xxxx}|$$

or $$\Rightarrow \|\vec{e}^{\,n}\|_\infty \le T\left(\frac{\Delta t}{2} K + \frac{\sigma \Delta x^2}{12} M\right) \xrightarrow[\Delta t \to 0,\ \Delta x \to 0,\ n \to \infty]{} 0$$

$$\|\vec{e}^{\,n+1}\|_\infty \le \underbrace{\|\vec{e}^{\,0}\|_\infty}_{=0} + (n+1)\tau\Delta t \le n\Delta t\,\tau + \Delta t\,\tau \le T\tau + \Delta t\,\tau$$

$$= (T+\Delta t)\tau \le (T+\underbrace{\Delta t}_{\to 0})\left(\frac{\Delta t}{2}K + \frac{\sigma \Delta x^2}{12} M\right) \to 0 \quad \text{and} \quad n\Delta t \le T$$

Remark: The inequality (*) is different for $i=1$ and $i=m$.

In fact, $r \le 1/2$.

For $i=1$: $\quad |e_1^{n+1}| \le r|\underbrace{e_0^n}_{=0}| + (1-2r)|e_1^n| + r|e_2^n| + \Delta t|\tau_1^n|$

For $i=m$: $\quad |e_m^{n+1}| \le r|e_{m-1}^n| + (1-2r)|e_m^n| + r|\underbrace{e_{m+1}^n}_{=0}| + \Delta t|\tau_m^n|$

In both cases:

$i=1$ or $i=m$: $\quad |e_i^{n+1}| \le (1-\underbrace{r}_{=1/2})\|\vec{e}^{\,n}\|_\infty + \Delta t\|\vec{\tau}^{\,n}\|_\infty \le \|e^n\|_\infty + \Delta t\|\vec{\tau}^{\,n}\|_\infty$

Same as (**). So, we proceed identically from here.

From PDE theory, we know that Solution at point P certainly depends on boundary data at Q and R.

From the previous graph, we conclude that the angle θ should be $\pi/2$ (or close to it) for Q and R to enter into the computation at P.

In previous experiments, we determine stability depending on "r" values for FT-CS Scheme:

a) $r = 10^{-1}$ for $\Delta x = 0.1$, $\Delta t = 10^{-3}$, $\sigma = 1$.

the num. scheme was stable.

$$r = \frac{\sigma \Delta t}{\Delta x^2}$$

In this case, $\theta = \tan^{-1}\left(\frac{\Delta x}{\Delta t}\right) = \tan^{-1}\left(\frac{10^{-1}}{10^{-3}}\right)$

$$= \boxed{\tan^{-1}(100) \simeq 1.56 \simeq \pi/2.}$$

b) $r = 1$, for $\Delta x = 0.1$, $\Delta t = 10^{-2}$, $\sigma = 1$

In this case,

$$\theta = \tan^{-1}\left(\frac{10^{-1}}{10^{-2}}\right) = \tan^{-1}(10) \simeq$$

$$\simeq 1.47 < \pi/2.$$

Numerical scheme is unstable.

c) If $r = 1/2$, and $\sigma = 1 \Rightarrow \Delta t = \frac{1}{2}\Delta x^2$

If $\Delta x = 0.1 \Rightarrow \theta = \tan^{-1}\left(\frac{\Delta x}{\frac{1}{2}\Delta x^2}\right) = \tan^{-1}\left(\frac{2}{\Delta x}\right) = \tan^{-1}\left(\frac{2}{10^{-1}}\right)$

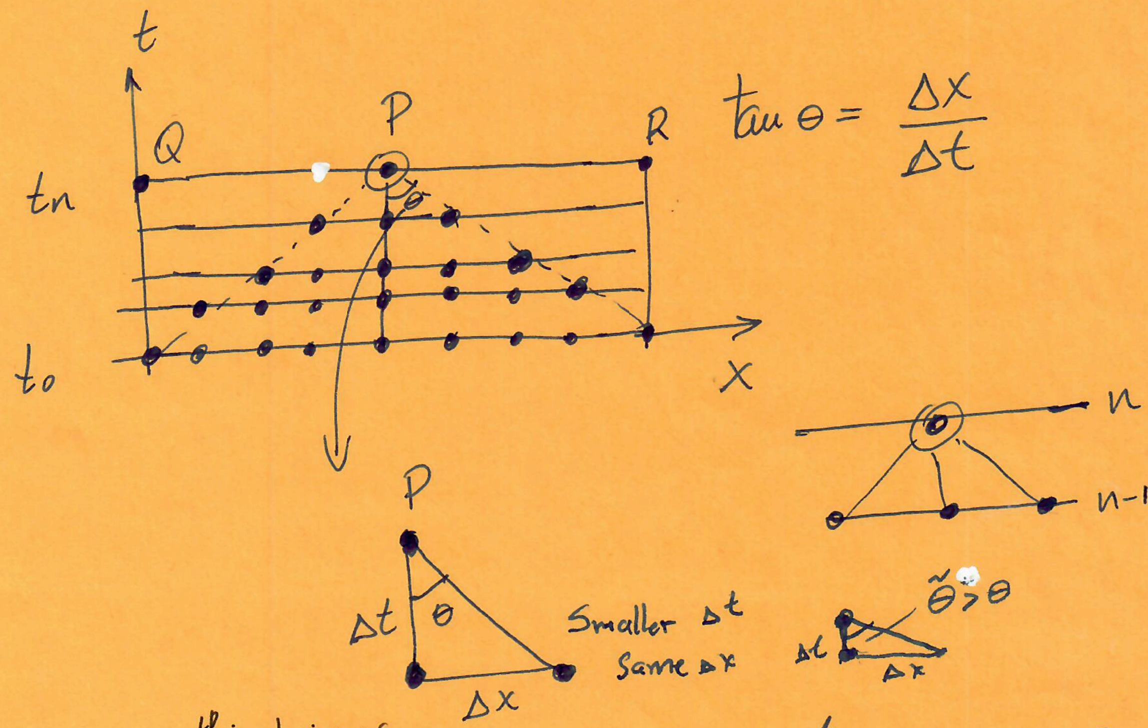
$$= \tan^{-1}(20) \simeq 1.5208 \simeq \pi/2.$$

Numerical scheme is stable

The condition $r \le 1/2$ impose limitations on the choice of Δt.

How can we define numerical schemes for our IBVP with less limitation on the choice of Δt?

<u>Idea</u>: Domain of dependence for num. sch. (2)



Obviously, for this choice of $\Delta x, \Delta t$ boundary values at points Q and R at level n don't enter into the computation of P at level n.

The previous analysis motivates the construction of implicit schemes. For implicit schemes, the solution at P will involve all the other unknowns at the same time level, and it will also include the boundary conditions at Q and R.

<u>Example:</u> BT-CS at the point (x_j, t_{n+1})

$$(u_t)_j^{n+1} = \sigma \, (u_{xx})_j^{n+1} \quad \text{vs} \quad \overset{FT-CS}{(u_t)_j^n = \sigma (u_{xx})_j^n}$$

approx. by
$$\frac{U_j^{n+1} - U_j^n}{\Delta t} = \sigma \, \frac{U_{j-1}^{n+1} - 2U_j^{n+1} + U_{j+1}^{n+1}}{\Delta x^2}$$

$j = 1, 2, \ldots, J-1$

Also called backward-Euler method.

$x_0 \quad x_1 \quad \ldots \quad x_{J-1} \quad x_J$

It can be written as

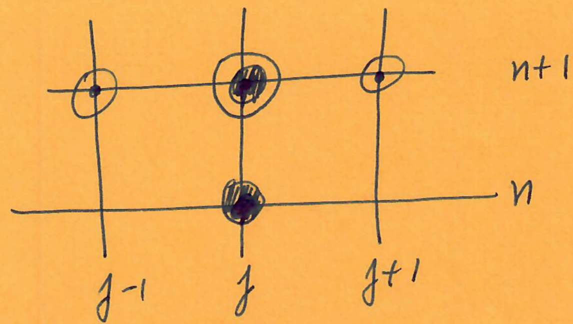
$$\boxed{-r\,U_{j-1}^{n+1} + (1+2r)\,U_j^{n+1} - r\,U_{j+1}^{n+1} = U_j^n,} \quad (4.1)$$

$j = 1, 2, \ldots, J-1$

For our IBVP (1), we also know

$$U_0^{n+1} = g(t_{n+1}) = g^{n+1}, \quad U_J^{n+1} = h^{n+1}$$

Computational stencil:

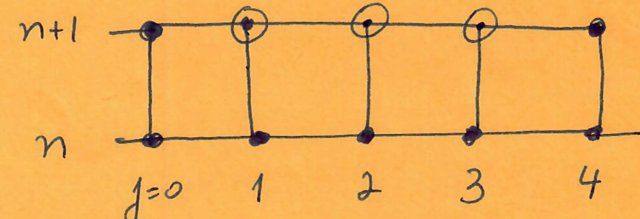


Obviously, for a given j equ. (4.1) is not enough. A System of equations needs to be Solved at every time level $n+1$.

$$\begin{cases} u_t = \sigma u_{xx}, & 0<x<1 \\ u(0,t) = g(t) \\ u(1,t) = h(t) \\ u(x,0) = \phi(x) \end{cases}$$

In particular, if $\underline{\underline{J=4}}$

Going from level t_n to t_{n+1}



We have a system of 3 equs. to be solved Simultaneously. In fact,

$$j=1, \quad -r \overset{BC}{U_0^{n+1}} + (1+2r)\, U_1 - r\, U_2^{n+1} = U_1^n$$

$$j=2, \quad -r\, U_1^{n+1} + (1+2r)\, U_2^{n+1} - r\, U_3^{n+1} = U_2^n$$

$$j=3, \quad -r\, U_2^{n+1} + (1+2r)\, U_3^{n+1} - r \overset{BC}{U_4^{n+1}} = U_3^n$$

With BCs:

$$U_0^{n+1} = g^{n+1}, \qquad U_4^{n+1} = h^{n+1}$$

The above system can be written in matrix form as

$$\begin{pmatrix} 1+2r & -r & 0 \\ -r & 1+2r & -r \\ 0 & -r & 1+2r \end{pmatrix} \begin{pmatrix} U_1^{n+1} \\ U_2^{n+1} \\ U_3^{n+1} \end{pmatrix} = \begin{pmatrix} U_1^n + r g^{n+1} \\ U_2^n \\ U_3^n + r h^{n+1} \end{pmatrix}$$

For a larger partition: $j = 1, \dots, J-1$
$n = 1, \dots, N$

A linear system for the unknowns: $U_1^n, U_2^n, \dots, U_{J-1}^n$ at each time level t_n needs to be solved.

$$A \vec{U}^{n+1} = \vec{F}^n$$

where

$$
\begin{array}{l}
j=1 \\ j=2 \\ \\ A = \\ \\ \\ j=J-1
\end{array}
\begin{bmatrix}
1+2r & -r & 0 & 0 & \cdots & 0 \\
-r & 1+2r & -r & 0 & \cdots & 0 \\
0 & \ddots & \ddots & \ddots & & \vdots \\
\vdots & & \ddots & \ddots & \ddots & 0 \\
0 & \cdots & & -r & 1+2r & -r \\
0 & \cdots & & & -r & 1+2r
\end{bmatrix}_{(J-1)\times(J-1)}
$$

$$
\vec{v}^{\,n+1} = \begin{bmatrix} v_1^{n+1} \\ v_2^{n+1} \\ \vdots \\ v_m^{n+1} \end{bmatrix}_{(J-1)\times 1}, \qquad
\vec{F}^{\,n} = \begin{bmatrix} v_1^n + r\overbrace{g^{n+1}}^{=r v_0^{n+1}} \\ v_2 \\ v_3 \\ \vdots \\ v_m^n + r\overbrace{h^{n+1}}^{=r v_J^{n+1}} \end{bmatrix}_{(J-1)\times 1}
$$

At every time step, this system is solved.