Another Implicit scheme:

Crank-Nicholson method.

$$\boxed{-\frac{r}{2}v_{i-1}^{n+1}+(1+r)v_i^{n+1}-\frac{r}{2}v_{i+1}^{n+1} = \frac{r}{2}v_{i-1}^{n}+(1-r)v_i^{n}+\frac{r}{2}v_{i+1}^{n}}$$

$i=1,2,..,m$

$v_0^n = g(t_n) = g^n$

$v_{m+1}^n = h(t_n) = h^n.$

Derivation:

Approximate u_t and u_{xx} at the point $(x_i, t_{n+1/2})$ using centered difference in both time and space with time step size $\frac{\Delta t}{2}$.

$$(u_t)_i^{n+1/2} = \sigma (u_{xx})_i^{n+1/2}$$

$$\Big\downarrow \ \text{CT-CS} \ \left(\tfrac{\Delta t}{2}\right) \ \Delta x$$

$$\frac{v_i^{n+1}-v_i^n}{2\left(\frac{k}{2}\right)} = \sigma\frac{v_{i+1}^{n+1/2}-2v_i^{n+1/2}+v_{i-1}^{n+1/2}}{h^2}$$

Using average in time

$$\frac{v_i^{n+1}-v_i^n}{k} = \sigma \frac{\frac{v_{i+1}^{n+1}+v_{i+1}^n}{2} - 2\,\frac{v_i^{n+1}+v_i^n}{2} + \frac{v_{i-1}^{n+1}+v_{i-1}^n}{2}}{h^2}$$

$$\Rightarrow \quad v_i^{n+1}-v_i^n = \frac{\sigma k}{2h^2}\left[v_{i+1}^{n+1}-2v_i^{n+1}+v_{i-1}^{n+1}+v_{i+1}^n-2v_i^n+v_{i-1}^n\right]$$

$$\xRightarrow{r=\frac{\sigma k}{h^2}} \quad \boxed{\begin{aligned} -\frac{r}{2}v_{i-1}^{n+1}+(1+r)v_i^{n+1}-\frac{r}{2}v_{i+1}^{n+1} &= \\ &= \frac{r}{2}v_{i-1}^n+(1-r)v_i^n+\frac{r}{2}v_{i+1}^n \end{aligned}} \qquad (11.0)$$

$i=1,\dots,m$

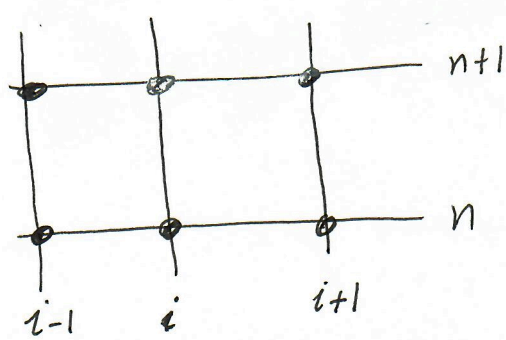
In matrix form:

due to BCs at $i=0$, and $i=m+1$.

$$\boxed{A\vec{v}^{\,n+1} = B\vec{v}^{\,n} + \vec{C}^n = \vec{D}^n} \qquad (11.1)$$

Remark:

Matrix Equation (11.1) needs to be solved at each time level "n".



Rmk: (11.0) can also be obtained as an average of

$$\frac{FT\text{-}CS + BT\text{-}CS}{2}.$$

where

$$A = \begin{bmatrix} 1+r & -r/2 & 0 & 0 & \cdots & 0 \\ -r/2 & 1+r & -r/2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & -r/2 & 1+r & -r/2 \\ 0 & \cdots & \cdots & \cdots & -r/2 & 1+r \end{bmatrix}_{m\times m}$$

$$B = \begin{bmatrix} 1-r & r/2 & 0 & 0 & \cdots & 0 \\ r/2 & 1-r & r/2 & 0 & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \ddots & 0 \\ \vdots & & & \ddots & \ddots & r/2 \\ 0 & \cdots & \cdots & \cdots & r/2 & 1-r \end{bmatrix}_{m\times m}$$

$$\vec{C}^n = \begin{array}{c} \begin{bmatrix} \frac{r}{2} g^{n+1}(x_0) + \frac{r}{2} g^n(x_0) \\ 0 \\ 0 \\ \vdots \\ \vdots \\ \frac{r}{2} h^{n+1}(x_{m+1}) + \frac{r}{2} h^n(x_{m+1}) \end{bmatrix}_{m\times 1} \end{array} \begin{array}{l} i=1 \\ \\ \\ \\ \\ i=m \end{array}, \qquad \vec{v}^n = \begin{bmatrix} v_1^n \\ v_2^n \\ \vdots \\ \vdots \\ v_m^n \end{bmatrix}_{m\times 1}$$

Weighted Average Scheme and Consistency of Crank-Nicholson scheme.

A more general scheme (family of schemes) Can be obtained by defining a Weighted average of forward and backward time differences with central spatial differences.

In fact, Consider $0 \le \theta \le 1$, then

$$(1-\theta) \times (FT\text{-}CS): \quad (1-\theta)\frac{U_i^{n+1} - U_i^n}{\Delta t} = \sigma(1-\theta)\frac{U_{i-1}^n - 2U_i^n + U_{i+1}^n}{(\Delta x)^2}$$

$$\theta \times (BT\text{-}CS): \quad \theta\frac{U_i^{n+1} - U_i^n}{\Delta t} = \sigma\theta\frac{U_{i-1}^{n+1} - 2U_i^{n+1} + U_{i+1}^{n+1}}{(\Delta x)^2}$$

Adding these two equations:

$$\boxed{\frac{U_i^{n+1} - U_i^n}{\Delta t} = \sigma\theta\frac{\delta^2 U_i^{n+1}}{(\Delta x)^2} + \sigma(1-\theta)\frac{\delta^2 U_i^n}{(\Delta x)^2}} \quad (13.1)$$

where
$$\boxed{\begin{aligned} &\delta U_i^n \overset{def}{=\!=} U_{i+1/2}^n + U_{i-1/2}^n \\ \Rightarrow \quad &\delta^2 U_i = \delta(\delta U_i) = U_{i-1}^n - 2U_i^n + U_{i+1}^n \end{aligned}} \quad (13.2)$$

(13.1) Can be rearranged as

$$U_i^{n+1} - \theta \frac{\sigma \Delta t}{\Delta x^2} \delta^2 U_i^{n+1} = U_i^n + (1-\theta) \frac{\sigma \Delta t}{\Delta x^2} \delta^2 U_i^n$$

or

$$\boxed{\begin{aligned} -\theta r U_{i-1}^{n+1} + (1+2\theta r) U_i^{n+1} - \theta r U_{i+1}^{n+1} \\ = (1-\theta) r U_{i-1}^n + \big(1 - 2(1-\theta) r\big) U_i^n + (1-\theta) r U_{i-1}^n \end{aligned}}$$

(13.3)

If $\theta = 1/2$

$$\boxed{\begin{aligned} -\frac{r}{2} U_{i-1}^{n+1} + (1+r) U_i^{n+1} - \frac{r}{2} U_{i+1}^{n+1} \\ = \frac{r}{2} U_i^n + (1-r) U_i^n + \frac{r}{2} U_{i-1}^n \end{aligned}} \qquad (13.4)$$

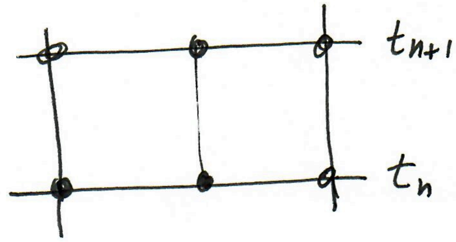
Crank-Nicholson.

If $\theta = 0 \Rightarrow$ We get FT-CS Explicit.

If $\theta = 1 \Rightarrow$ " " BT-CS implicit.

If $\theta = 1/2 \Rightarrow$ " " Crank-Nicholson Implicit.

If $0 < \theta < 1 \Rightarrow$ " " Implicit methods.



Stencil for $0 < \theta < 1$.

Notice,

i) $\theta = 0 \Rightarrow$ FT-CS. Explicit

ii) $\theta = 1 \Rightarrow$ BT-CS. Implicit Backward Euler.

iii) $\theta = 1/2 \Rightarrow$ Crank-Nicholson. Implicit.

Consistency and Local Truncation Error for Crank-Nicholson.

$$-\frac{r}{2} v_{i-1}^{n+1} + (1+r) v_i^{n+1} - \frac{r}{2} v_{i+1}^{n+1}$$

$$- \frac{r}{2} v_{i-1}^{n} - (1-r) v_i^n - \frac{r}{2} v_{i+1}^n = 0$$

or

$$-\frac{r}{2}\left[v_{i-1}^{n+1} - 2v_i^{n+1} + v_{i+1}^{n+1}\right] + v_i^{n+1} +$$

$$-\frac{r}{2}\left[v_{i-1}^{n} - 2v_i^{n} + v_{i+1}^{n}\right] - v_i^n = 0 \qquad (14.1)$$

We define

$$\boxed{\begin{aligned} P_\Delta v_i^n \stackrel{def}{=\!=} & -\frac{r}{2}\left[v_{i-1}^{n+1} - 2v_i^{n} + v_{i+1}^{n+1}\right] + v_i^{n+1} \\ & - \frac{r}{2}\left[v_{i-1}^{n} - 2v_i^{n} + v_{i+1}^{n}\right] - v_i^n \end{aligned}} \qquad (14.2)$$

Consistency Crank-Nicholson

Subst. of $u(x_i, t_n)$ into (14.2)

$$P_\Delta u_i^n = -\frac{r}{2}\underbrace{\left[u_{i-1}^{n+1} - 2u_i^{n+1} + u_{i+1}^{n+1}\right]}_{A} + u_i^{n+1} - \frac{r}{2}\overbrace{\left[u_{i-1}^{n} - 2u_i^{n} + u_{i+1}^{n}\right]}^{B} - u_i^n .$$

Using Taylor expansions $\Rightarrow$

$$= -\frac{r\Delta x^2}{2}\left[\underbrace{(u_{xx})_i^{n+1}}_{C} + \underbrace{\frac{\Delta x^2}{12}(u_{4x})_i^{n+1}}_{D} + O(\Delta x^4)\right] + \underbrace{\left(u_i^{n+1} - u_i^n\right)}_{E}$$

$$-\frac{r\Delta x^2}{2}\left[(u_{xx})_i^n + \frac{\Delta x^2}{12}(u_{4x})_i^n + O(\Delta x^4)\right] \rightarrow B$$

Then,

$$P_\Delta u_i^n = -\frac{r\Delta x^2}{2}\Bigg\{ \overbrace{(u_{xx})_i^n + \Delta t\,(u_{xxt})_i^n + \frac{\Delta t^2}{2}\boxed{(u_{xxtt})_i^n} + O(\Delta t^3)}^{C}$$

$$+ \overbrace{\frac{\Delta x^2}{12}\left[(u_{4x})_i^n + \Delta t\,(u_{4xt})_i^n + \frac{\Delta t^2}{2}(u_{4xtt})_i^n + O(\Delta t^3)\right]}^{D} + O(\Delta x^4)\Bigg\}$$

$$+ \overbrace{\left(\Delta t\,(u_t)_i^n + \frac{\Delta t^2}{2}(u_{tt})_i^n + \frac{\Delta t^3}{3!}\boxed{(u_{3t})_i^n} + O(\Delta t^4)\right)}^{E}$$

$$-\frac{r\Delta x^2}{2}\overbrace{\left[(u_{xx})_i^n + \frac{\Delta x^2}{12}(u_{4x})_i^n + O(\Delta x^4)\right]}^{B}$$

Considering

using heat equation

$$(u_{xx})_t = \frac{u_{tt}}{\sigma}, \qquad u_{ttxx} = \frac{u_{ttt}}{\sigma}$$

then,

$$P_\Delta u_i^n = -\frac{r\Delta x^2}{2}\Big\{ 2(u_{xx})_i^n + \frac{1}{\sigma}(u_{tt})_i^n \Delta t + \frac{1}{\sigma}(u_{ttt})_i^n \frac{\Delta t^2}{2} + O(\Delta t^3)$$
$$+ \frac{\Delta x^2}{12}\Big[2(u_{4x})_i^n + \Delta t\,(u_{4xt})_i^n + \frac{\Delta t^2}{2}(u_{4xtt})_i^n + O(\Delta t^3)\Big] + O(\Delta x^4)\Big\}$$
$$+ \Delta t\,(u_t)_i^n + \frac{\Delta t^2}{2}(u_{tt})_i^n + \frac{\Delta t^3}{3!}(u_{3t})_i^n + O(\Delta t^4) = 0$$

Using $r = \frac{\sigma \Delta t}{\Delta x^2}$ results

$$P_\Delta u_i^n = \Delta t\Big[\boxed{-\sigma(u_{xx})_i^n} - \cancel{(u_{tt})_i^n \frac{\Delta t}{2}} - (u_{3t})_i^n \frac{\Delta t^2}{4} + O(\Delta t^3)$$
$$-\sigma(u_{4x})_i^n \frac{\Delta x^2}{12} - \sigma(u_{4xt})_i^n \frac{\Delta t}{2}\frac{\Delta x^2}{12} - \sigma(u_{4xtt})_i^n \frac{\Delta t^2}{4}\frac{\Delta x^2}{12}$$
$$+ O(\Delta t^3) + O(\Delta x^4) + \boxed{(u_t)_i^n} + \cancel{\frac{\Delta t}{2}(u_{tt})_i^n} + \frac{\Delta t^2}{3!}(u_{3t})_i^n$$
$$+ O(\Delta t^3)\Big]$$

$$= \Delta t\Big[(u_t)_i^n - \sigma(u_{xx})_i^n - \frac{\Delta t^2}{12}(u_{3t})_i^n - \sigma(u_{4x})_i^n \frac{\Delta x^2}{12}$$
$$- \frac{\sigma}{48}(u_{4xtt})_i^n \Delta t\,\Delta x^2$$
$$+ O(\Delta x^4) + O(\Delta x^2 \Delta t^2)\Big]$$

Therefore,

$$\boxed{\begin{aligned}\frac{P_\Delta u_i^n}{\Delta t} - \left((u_t)_i^n - \sigma (u_{xx})_i^n\right) &= \frac{-1}{12}(u_{3t})_i^n \Delta t^2 \\ &\quad - \frac{\sigma}{12}(u_{4x})_i^n \Delta x^2 - \frac{\sigma}{48}(u_{4xtt})_i^n \Delta t \Delta x^2 \\ &\quad + O(\Delta x^4) + O(\Delta x^2 \Delta t^2). \\ &= \tau_i^n \text{ (Local Truncation error)}\end{aligned}}$$

Since the v_i^n soln. of

$\frac{P_\Delta v_i^n}{\Delta t} = 0$ is the same of $P_\Delta v_i^n = 0$

then C-N is $O(\Delta t^2) + O(\Delta x^2)$.

Linear System for Weighted Average Scheme

The discrete equation is given by (13.3).

$$\boxed{-r\theta U_{i-1}^{n+1} + (1+2r\theta)U_i^{n+1} - r\theta U_{i+1}^{n+1} = r(1-\theta)U_{i-1}^{n} + \left(1-2r(1-\theta)\right)U_i^{n} + r(1-\theta)U_{i+1}^{n}} \qquad (18.2)$$

$i = 1 .. m$

B/C's: $U_0^n = f^n$, $U_{m+1}^n = \;{}^n$, $n = 1, 2, \ldots, N_T$.

I.C: $U^0 = \phi_i$

Linear syst. in the next page.

$j=1$ $$\underbrace{-r\theta U_0^{n+1}}_{=-r\theta f^{n+1}} + (1+2r\theta)U_1^{n+1} - r\theta U_2^{n+1} =$$
$$= \underbrace{r(1-\theta)U_0^n}_{=r(1-\theta)f^n} + (1-2r(1-\theta))U_1^n + r(1-\theta)U_2^n$$

$j=2$ $$-r\theta U_1^{n+1} + (1+2r\theta)U_2^{n+1} - r\theta U_3^{n+1} =$$

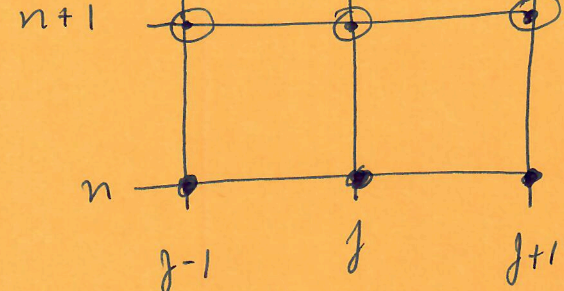
$\vdots$

$j=J-1$ $$-r\theta U_{J-2}^{n+1} + (1+2r\theta)U_{J-1}^{n+1} \underbrace{-r\theta U_J^n}_{=-r\theta g^{n+1}} = r(1-\theta)U_{J-2}^n$$
$$+ (1-2r(1-\theta))U_{J-1}^n + \underbrace{r(1-\theta)U_J^n}_{=r(1-\theta)g^n}$$

$$\therefore \left(\begin{bmatrix} 1 & & & \\ & 1 & & 0 \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} + r\theta \begin{bmatrix} 2 & -1 & 0 & 0 & \cdots & 0 \\ -1 & 2 & -1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & & -1 & 2 & -1 \\ & & & & -1 & 2 \end{bmatrix} \right) \begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{J-1}^{n+1} \end{bmatrix} =$$

$$= \left(\begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} + r(1-\theta) \begin{bmatrix} -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & & \vdots \\ & & & & & & 1 \\ & & & & & 1 & -2 \end{bmatrix} \right) \begin{bmatrix} U_1^n \\ U_2^n \\ \vdots \\ U_{J-1}^n \end{bmatrix} + r \begin{bmatrix} \theta f^{n+1} + (1-\theta) f^n \\ 0 \\ \vdots \\ 0 \\ \theta g^{n+1} + (1-\theta) g^n \end{bmatrix} \qquad (19.1)$$

For $\theta \in (0,1)$ the computational stencil looks like



(.) in matrix form is also a tridiag. non-homogeneous system for $\theta \in (0,1)$. It can be written in more compact form as

$$\boxed{\left[I - r\theta C\right]\vec{U}^{n+1} = \left[I + r(1-\theta)C\right]\vec{U}^{n} + r\vec{f}^{n},} \qquad (20.1)$$

where

$$\vec{U}^n = \begin{bmatrix} U_1^n \\ U_2^n \\ U_3^n \\ \vdots \\ U_{J-1}^n \end{bmatrix}, \qquad C = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & \cdots & 0 \\ & \ddots & \ddots & \ddots & 1 \\ 0 & & & 1 & -2 \end{bmatrix}$$

$$\vec{f}^n = \begin{bmatrix} \theta f^{n+1} + (1-\theta) f^n \\ 0 \\ \vdots \\ 0 \\ \theta g^{n+1} + (1-\theta) g^n \end{bmatrix}$$