Review: Iterative Solution of Linear Systems

$$\boxed{A \quad \vec{x} = \vec{b}}$$

$$\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$$

$$D = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \quad L = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

Jacobi Iteration:

$$\boxed{A = D - L - U}$$

Start with guess

$$\vec{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}$$

$$\boxed{\begin{aligned} 2x_1^{(k)} &= \left(x_2^{(k-1)} + 1\right) \\ 3x_2^{(k)} &= \left(x_1^{(k-1)} + x_3^{(k-1)} + 8\right) \\ 2x_3^{(k)} &= \left(x_2^{(k-1)} - 5\right) \end{aligned}}$$

$$D \quad \vec{x}^{k} = L+U \quad \vec{x}^{(k-1)} \quad \vec{b}$$

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{pmatrix} + \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$$

Assuming D is invertible

$$\boxed{\vec{x}^{(k)} = D^{-1}(L+U)\vec{x}^{(k-1)} + D^{-1}\vec{b}}$$

$$\vec{X}^{(k)} = \underbrace{D^{-1}(L+U)}\,\vec{x}^{(k-1)} + \underbrace{D^{-1}}\vec{b}$$

$$\begin{pmatrix} X_1^{(k)} \\ X_2^{(k)} \\ X_3^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} X_1^{(k-1)} \\ X_2^{(k-1)} \\ X_3^{(k-1)} \end{pmatrix} + \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix}$$

$$\boxed{\vec{X}^{(k)} = T_J \quad \vec{X}^{(k-1)} + \vec{C}}$$

Remark: A may be invertible, but D^{-1} does not exist.

In fact,

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix}, \quad \det A = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{vmatrix} = 1 \begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix} = -1 \neq 0$$

Then A is invertible, but

$$D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ is not invertible.}$$

Gauss-Siedel Iteration:

$$\boxed{\begin{aligned} x_1^{(k)} &= \frac{1}{2}\left(x_2^{(k-1)} + 1\right) \\ x_2^{(k)} &= \frac{1}{3}\left(x_1^{(k)} + x_3^{(k-1)} + 8\right) \\ x_3^{(k)} &= \frac{1}{2}\left(x_2^{(k)} - 5\right) \end{aligned}}$$

Equivalent to

$$\begin{aligned} x_1^{(k)} \qquad\qquad &= x_2^{(k-1)} + 1 \\ -x_1^{(k)} + 3x_2^{(k)} \qquad &= x_3^{(k-1)} + 8 \\ -x_2^{(k)} + 2x_3^{(k)} &= -5 \end{aligned}$$

$$\begin{pmatrix} 1 & 0 & 0 \\ -1 & 3 & 0 \\ 0 & -1 & 2 \end{pmatrix} \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{pmatrix} + \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$$

$$(D-L)\,\vec{x}^{(k)} = U\,\vec{x}^{(k-1)} + \vec{b}$$

Assuming $D-L$ is invertible

$$\boxed{\vec{x}^{(k)} = (D-L)^{-1}U\,\vec{x}^{(k-1)} + (D-L)^{-1}\vec{b}}$$

or

$$\boxed{\vec{x}^{(k)} = T_g\,\vec{x}^{(k-1)} + \vec{C}_g.}$$

where $T_g = (D-L)^{-1}U$, $\vec{C}_g = (D-L)^{-1}\vec{b}$.

SOR

1) Apply Gauss-Seidel and obtain $\vec{X}^{(k)}$

2) Update $\vec{X}^{(k)}$ by a new $X^{(k)}$ by using the linear combination:

$$\boxed{X_i^{(k)} = \omega X_i^{GS(k)} + (1-\omega) X_i^{(k-1)}}, \quad k=1,2,\dots \tag{4.1}$$

Where $\omega \in (0,2)$

For Computation

$$\boxed{\begin{aligned} X_1^{(k)} &= \omega\left[\tfrac{1}{2} X_2^{(k-1)} + \tfrac{1}{2}\right] + (1-\omega) X_1^{(k-1)} \\ X_2^{(k)} &= \omega\left[\tfrac{1}{3} X_1^{(k)} + \tfrac{1}{3} X_3^{(k-1)} + 8/3\right] + (1-\omega) X_2^{(k-1)} \\ X_3^{(k)} &= \omega\left[\tfrac{1}{2} X_2^{(k)} - \tfrac{5}{2}\right] + (1-\omega) X_3^{(k-1)} \end{aligned}} \tag{4.2}$$

or

$$\boxed{\begin{aligned} 2X_1^{(k)} &= \omega\left[X_2^{(k-1)} + 1\right] + (1-\omega) 2 X_1^{(k-1)} \\ 3X_2^{(k)} &= \omega\left[X_1^{(k)} + X_3^{(k-1)} + 8\right] + (1-\omega) 3 X_2^{(k-1)} \\ 2X_3^{(k)} &= \omega\left[X_2^{(k)} - 5\right] + (1-\omega) 2 X_3^{(k-1)} \end{aligned}} \tag{4.3}$$

$$\left[\underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{=D} - \omega \underbrace{\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}_{=L}\right] \begin{pmatrix} x_1^{(k)} \\ x_2^{(k)} \\ x_3^{(k)} \end{pmatrix}$$

$$= \left[\omega \underbrace{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}}_{=U} + (1-\omega) \underbrace{\begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{pmatrix}}_{=D}\right] \begin{pmatrix} x_1^{(k-1)} \\ x_2^{(k-1)} \\ x_3^{(k-1)} \end{pmatrix} + \omega \underbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{=I} \begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}$$

or

$$\boxed{(D - \omega L)\vec{x}^{(k)} = (\omega U + (1-\omega) D)\,\vec{x}^{(k-1)} + \omega I \vec{b}.} \qquad (5.1)$$

Assuming $D-\omega L$ is invertible

Then,

$$\vec{x}^{(k)} = \Big[\underbrace{(D-\omega L)^{-1}(\omega U + (1-\omega) D)}_{=T_\omega}\Big]\,\vec{x}^{(k-1)} + \underbrace{\omega I \vec{b}}_{=\vec{c}_\omega}.$$

or

$$\boxed{\vec{x}^{(k)} = T_\omega \vec{x}^{(k-1)} + \vec{c}_\omega} \qquad (5.2)$$