# 6.5 Iterative Solution of Linear equations.

(Burden-Faires)

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 3x_2 - x_3 = 8 \\ -x_2 + 2x_3 = -5 \end{cases} \iff A\vec{x} = \vec{b} \qquad (1.1)$$

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{=\vec{x}} = \underbrace{\begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}}_{=\vec{b}} \iff A\vec{x} = \vec{b}$$

$$\begin{cases} x_1 = \frac{1}{2}x_2 + \frac{1}{2} \\ x_2 = \frac{1}{3}x_1 + \frac{1}{3}x_3 + \frac{8}{3} \\ x_3 = \frac{1}{2}x_2 - \frac{5}{2} \end{cases} \qquad (1.2)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix}$$

$$\vec{x} = T\vec{x} + \vec{c} \qquad (1.3)$$

We are replacing $A\vec{x} = \vec{b}$ by the equivalent System $\vec{x} = T\vec{x} + \vec{c}$

An alternative way to look at the previous linear system is as

$$\left[\overset{=D}{\begin{pmatrix}2&0&0\\0&3&0\\0&0&2\end{pmatrix}} - \overset{=L}{\begin{pmatrix}0&0&0\\+1&0&0\\0&+1&0\end{pmatrix}} - \overset{=U}{\begin{pmatrix}0&+1&0\\0&0&+1\\0&0&0\end{pmatrix}}\right]\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix} = \begin{pmatrix}1\\8\\-5\end{pmatrix} \qquad (1.4)$$

or briefly

$$A\vec{x} = \vec{b} \iff (D-L-U)\vec{x} = \vec{b}.$$

We can now write: $D\vec{x} = (L+U)\vec{x} + \vec{b}$. $\qquad$ (1.5)

In our case,

$$\begin{cases} 2x_1 = x_2 + 1 \\ 3x_2 = x_1 + x_3 + 8. \\ 2x_3 = x_2 - \frac{5}{2} \end{cases}$$

where

$$D = \begin{pmatrix}2&0&0\\0&3&0\\0&0&2\end{pmatrix}$$

$$L = \begin{pmatrix}0&0&0\\1&0&0\\0&1&0\end{pmatrix}$$

$$U = \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$$

Now, $D$ is an invertible matrix because

$$\det(D) = 12 \neq 0.$$

Then, we can write (1.5) as

$$\vec{x} = D^{-1}(L+U)\vec{x} + D^{-1}\vec{b}. \qquad (1.6)$$

which in our example corresponds to system (1.3). (1.2)

Therefore, if we define

$$\boxed{T = D^{-1}(L+U)} \quad \text{and} \quad \boxed{\vec{c} = D^{-1}\vec{b}}$$

(1.6) reduces to

$$\boxed{\vec{x} = T\vec{x} + \vec{c}}$$

Same as (1.3).

This is the equation used by Jacobi to obtain an approximation to the solution $\vec{x}$ of the linear system

$$A\vec{x} = \vec{b}$$

iterating.

So,

for our example,
$$T = D^{-1} \overset{L+U}{\overset{\shortparallel}{\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}}}$$

Then,

$$T = \begin{pmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix}$$

and $\boxed{\vec{x} = T\vec{x} + \vec{c}}$,

Where
$$T = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix}$$

## Jacobi Iteration:

Start with an arbitrary initial approximation

$$\vec{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}$$

and follow with the iterative formula:

$$\vec{x}^{(k)} = T\,\vec{x}^{(k-1)} + \vec{c}$$

$$\begin{cases} x_1^{(k)} = \frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \\ x_2^{(k)} = \frac{1}{3} x_1^{(k-1)} + \frac{1}{3} x_3^{(k-1)} + \frac{8}{3} \\ x_3^{(k)} = \frac{1}{2} x_2^{(k-1)} - \frac{5}{2} \end{cases}$$

Starting with $\vec{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

we find $\vec{x}^{(1)} = (0.5,\ 2.66,\ -2.5)^T$

$\vdots$

$\vec{x}^{(21)} = (2, 3, -1)^T$ exact soln.

In general,

$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1i}x_i + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2i}x_i + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n = b_i \\ \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{ni}x_i + \cdots + a_{nn}x_n = b_n \end{cases}$$

is equivalent to

$$a_{11}x_1 = -a_{12}x_2 - \cdots - a_{1i}x_i - \cdots - a_{1n}x_n + b_1$$

$$a_{22}x_2 = -a_{21}x_1 - a_{23}x_3 - \cdots - a_{2i}x_i - \cdots - a_{2n}x_n + b_2$$

$$\vdots$$

$$a_{ii}x_i = -a_{i1}x_1 - \cdots - a_{i,i-1}x_{i-1} - a_{i,i+1}x_{i+1} - \cdots - a_{in}x_n + b_i$$

$$\vdots$$

$$a_{nn}x_n = -a_{n1}x_1 - \cdots - a_{n,i}x_i - \cdots - a_{n,n-1}x_{n-1} + b_n$$

or in more compact form

$$a_{ii}x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j + b_i, \qquad i = 1, \ldots, n. \quad (3.1)$$

or $D\vec{x} = (L+U)\vec{x} + \vec{b}.$

If $a_{ii} \neq 0$

$$x_i = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j + b_j\right] \qquad (4.1)$$

$$i=1,\dots,n$$

or in matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = -\begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1i}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \cdots & \frac{a_{2i}}{a_{22}} & \cdots & \frac{a_{2n}}{a_{22}} \\ \vdots & & & & & & \\ \frac{a_{i1}}{a_{ii}} & \frac{a_{i2}}{a_{ii}} & \cdots & \frac{a_{i,i-1}}{a_{ii}} & 0 & \frac{a_{i,i+1}}{a_{ii}} \cdots & \frac{a_{in}}{a_{ii}} \\ \vdots & & & & & & \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \cdots & & \frac{a_{ni}}{a_{nn}} & \cdots \frac{a_{n,n-1}}{a_{nn}} & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_i/a_{ii} \\ \vdots \\ b_n/a_{nn} \end{pmatrix}$$

or

$$\vec{x} = T\vec{x} + \vec{c} \quad \text{or} \quad \vec{x} = D^{-1}(L+U)\vec{x} + D^{-1}\vec{b} \qquad (4.2)$$

## Comments about existence and uniqueness of Solutions.

i) The matrix $A$ in (1.1) is strictly diagonally dominant then, (1.1) has a unique soln.

ii) Also, because $a_{ii} \neq 0$, $i = 1, 2, 3$. (equivalent to $D^{-1}$ exists)
$T_j$ can be formed and the two systems

$$A\vec{x} = \vec{b} \quad \text{and} \quad \vec{x} = T_j\vec{x} + \vec{C}$$

are equivalent. It means that they have the same solutions.

iii) Therefore, (1.1) and (1.3) have the same unique solution.

iv) Also, notice

$$\|T\|_\infty = \frac{1}{3} + \frac{1}{3} = \frac{2}{3} < 1.$$

We will prove later that $\|T\| < 1$ is a sufficient condition for the convergence of a general iterative technique:

$$\vec{x}^{(k)} = \hat{T}\,\vec{x}^{(k-1)} + \vec{C}.$$

## Application to our example:

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}}_{A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{\vec{x}} = \underbrace{\begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}}_{\vec{b}}$$

<u>Jacobi's method</u> leads to (Dividing matrix A by 3 and splitting this matrix in two.)

$$\left[\overbrace{\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}^{=I} + \overbrace{\begin{pmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{pmatrix}}^{=-T}\right] \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix}$$

$$(I - T)\,\vec{x} = \vec{c}$$

Eigenvalues of matrix $T$.

$$0 = |T - \lambda I| = \begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/3 & -\lambda & 1/3 \\ 0 & 1/2 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{2}{6}\lambda = \lambda\left(-\lambda^2 + \frac{1}{3}\right)$$

$$\lambda_1 = 0, \quad \lambda_2 = \frac{1}{\sqrt{3}}, \quad \lambda_3 = -\frac{1}{\sqrt{3}}$$

Then, $\rho(T) = \frac{1}{\sqrt{3}}$

We will prove later : $\rho(T) < 1 \Rightarrow$ Jacobi converges

## Jacobi Iteration:

$$\vec{x}^{(k)} = T\,\vec{x}^{(k-1)} + \vec{c} \qquad (5.1)$$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}\, x_j^{(k-1)} + b_i\right] \qquad (5.2)$$

$i = 1, .., n$

## Gauss-Siedel iteration

In our previous example,

$$\begin{cases} x_1^{(k)} = \frac{1}{2}\, x_2^{(k-1)} + \frac{1}{2} \\ x_2^{(k)} = \frac{1}{3}\, x_1^{(k)} + \frac{1}{3}\, x_3^{(k-1)} + \frac{8}{3} \\ x_3^{(k)} = \frac{1}{2}\, x_2^{(k)} - \frac{5}{2} \end{cases} \qquad (5.3)$$

Starting with $\vec{x}^{(0)} = (0, 0, 0)$.

$$\vec{x}^{(1)} = (0.5,\ 2.8,\ -1.08)^T$$

$$\vec{x}^{(2)} = (1.9,\ 2.9,\ -1.02)^T$$

$$\vdots$$

$$\vec{x}^{(9)} = (2, 3, -1)^T$$

(5')

From (5.3)

$$\boxed{X_1^{(k)} = \frac{1}{2} X_2^{(k-1)} + \frac{1}{2}.}$$

$$X_2^{(k)} = \frac{1}{3} X_1^{(k)} + \frac{1}{3} X_3^{(k-1)} + \frac{8}{3} =$$

$$= \frac{1}{3}\left[\frac{1}{2} X_2^{(k-1)} + \frac{1}{2}\right] + \frac{1}{3} X_3^{(k-1)} + \frac{8}{3} =$$

$$\therefore \boxed{X_2^{(k)} = \frac{1}{6} X_2^{(k-1)} + \frac{1}{3} X_3^{(k-1)} + \frac{17}{6}}$$

$$X_3^{(k)} = \frac{1}{2} X_2^{(k)} - \frac{5}{2} = \frac{1}{2}\left[\frac{1}{6} X_2^{(k-1)} + \frac{1}{3} X_3^{(k-1)} + \frac{17}{6}\right] - \frac{5}{2}$$

$$\therefore \boxed{X_3^{(k)} = \frac{1}{12} X_2^{(k-1)} + \frac{1}{6} X_3^{(k-1)} - \frac{13}{12}}$$

Therefore,

$$\vec{X}^{(k)} = T_g \vec{x}^{(k-1)} + \vec{C}_g,$$

where

$$T_g = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ 0 & 1/6 & 1/3 \\ 0 & 1/12 & 1/6 \end{pmatrix}, \quad \vec{C}_g = \begin{pmatrix} 1/2 \\ 17/6 \\ -13/12 \end{pmatrix}$$

A more general way to look at Gauss-Siedel:

$$\begin{cases} 2x_1^{(k)} = x_2^{(k-1)} + 1 \\ -x_1^{(k)} + 3x_2^{(k)} = x_3^{(k-1)} + 8 \\ -x_2^{(k)} + 2x_3^{(k)} = -5 \end{cases}$$

or $(D-L)\vec{x}^{(k)} = U\vec{x}^{(k-1)} + \vec{b}$

$$\Rightarrow \boxed{\vec{x}^{(k)} = (D-L)^{-1}U\vec{x}^{(k-1)} + (D-L)^{-1}\vec{b}.}$$

or $\boxed{\vec{x}^{(k)} = T_g\vec{x}^{(k-1)} + \vec{c}_g}$

where $\boxed{T_g = (D-L)^{-1}U}$ and $\boxed{\vec{c}_g = (D-L)^{-1}\vec{b}.}$

$A\vec{x}=\vec{b}$

In General for any linear system, Gauss- Seidel Algorithm Can be written as (A nonsingular)

$$
\begin{cases}
a_{11}x_1^{(k)} = -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \dots - a_{1n}x_n^{(k-1)} + b_1 \\
a_{21}x_1^{(k)} + a_{22}x_2^{(k)} = -a_{23}x_3^{(k-1)} - \dots - a_{2n}x_n^{(k-1)} + b_2 \\
\vdots \\
a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \dots + a_{ii}x_i^{(k)} = -a_{i,i+1}x_{i+1}^{(k-1)} + \dots + a_{i,n}x_n^{(k-1)} + b_i \\
\vdots \\
a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{nn}x_n^{(k)} = \dots + b_n
\end{cases}
$$

$$
\boxed{x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right] + \frac{b_i}{a_{ii}}}, \quad i=1,2,..,n
$$

$\hookrightarrow$ Formula used for computation.

Theoretical formula:

$$
\boxed{(D-L)\,\vec{x}^{(k)} = U\vec{x}^{(k-1)} + \vec{b}.}
$$

$$
\boxed{\begin{aligned}\vec{x}^{(k)} &= (D-L)^{-1}U\,\vec{x}^{(k-1)} + (D-L)^{-1}\vec{b} \\ \text{or} \quad \vec{x}^{(k)} &= T_g\vec{x}^{(k-1)} + \vec{C}_g\end{aligned}}
$$

Summarizing Gauss-Seidel.

$$a_{ii} x_i^{(k)} = - \sum_{\substack{j=1 \\ j<i}}^{n} a_{ij} x_j^{(k)} - \sum_{\substack{j=1 \\ j>i}}^{n} a_{ij} x_j^{(k-1)} + b_i \quad (6.1)$$

or

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[ - \sum_{\substack{j=1 \\ j<i}}^{n} a_{ij} x_j^{(k)} - \sum_{\substack{j=1 \\ j>i}}^{n} a_{ij} x_j^{(k-1)} + b_i \right] \quad (6.2).$$

$$i = 1, \dots, n$$

## Succesive Overrelaxation (SOR)

Once $\vec{x}_i^{(k)}$ has been obtained from Gauss-Siedel iteration. The convergence can be accelerated computing a new $x_i^{(k)}$ by means

$$x_i^{(k)} = \omega x_i^{(k)} + (1-\omega) x_i^{(k-1)} \qquad (6.3)$$

Where $0 < \omega < 2$

Applying (6.3) to our example. It means
Combining (6.3) and (5.3).

$$\begin{cases} x_1^{(k)} = \omega\left[\frac{1}{2}x_2^{(k-1)} + \frac{1}{2}\right] + (1-\omega)\,x_1^{(k-1)} \\ x_2^{(k)} = \omega\left[\frac{1}{3}x_1^{(k)} + \frac{1}{3}x_3^{(k-1)} + \frac{8}{3}\right] + (1-\omega)\,x_2^{(k-1)} \\ x_3^{(k)} = \omega\left[\frac{1}{2}x_2^{(k)} - \frac{5}{2}\right] + (1-\omega)\,x_3^{(k-1)} \end{cases}$$

In general,

$$x_i^{(k)} = \frac{\omega}{a_{ii}}\left[-\sum_{\substack{j=1\\ j<i}}^{n} a_{ij}\,x_j^{(k)} - \sum_{\substack{j=1\\ j>i}}^{n} a_{ij}\,x_j^{(k-1)} + b_i\right] + (1-\omega)\,x_i^{(k-1)} \qquad (7.1)$$

$i = 1, \ldots, n.$

starting with $x^{(0)} = (0,0,0)^T$ in our example and $\boxed{\omega = 1.1}$

$\vec{x}^{(1)} = (0.55,\ 3.135,\ -1.026)^T$

$\vdots$

$\vec{x}^{(7)} = (2,\ 3,\ -1)$ exact soln.

# SOR method in matrix form

$$x_i^{(k)} = \frac{\omega}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + b_i\right] + (1-\omega)x_i^{(k-1)}$$

Then,

$$a_{ii}x_i^{(k)} = -\omega\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \omega\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + \omega b_i + (1-\omega)a_{ii}x_i^{(k-1)}$$

or

$$a_{ii}x_i^{(k)} + \omega\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} = -\omega\sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)} + \omega b_i + (1-\omega)a_{ii}x_i^{(k-1)}$$

In matrix form in terms of D, L, U.

$$(D-\omega L)\vec{x}^{(k)} = \omega U\vec{x}^{(k-1)} + (1-\omega)D\vec{x}^{(k-1)} + \omega I\vec{b}$$

Therefore,

$$\boxed{\vec{x}^{(k)} = \overbrace{(D-\omega L)^{-1}\left[\omega U + (1-\omega)D\right]}^{=T_\omega}\vec{x}^{(k-1)} + \overbrace{\omega(D-\omega L)^{-1}\vec{b}}^{=\vec{c}_\omega}.}$$

or $\boxed{\vec{x}^{(k)} = T_\omega\vec{x}^{(k-1)} + \vec{c}_\omega}$

## Summarizing.

Popular iterative algorithm used to approximate

$$A\vec{x} = \vec{b}, \quad \text{where } A = D - L - U.$$

$D$ → diag.   $L$ → Lower Triang   $U$ → upper triang.

Can be written as

### I. Jacobi Method:

$$D\vec{x}^{(k)} = (L+U)\vec{x}^{(k-1)} + \vec{b}.$$

$Q = D$

### II. Gauss-Seidel Method:

$$(D-L)\vec{x}^{(k)} = U\vec{x}^{(k-1)} + \vec{b}.$$

$Q = D-L$

### III. SOR METHOD:

$$(D-\omega L)\vec{x}^{(k)} = [\omega U + (1-\omega)D]\vec{x}^{(k-1)} + \omega\vec{b}$$

or

$$\left(\frac{D}{\omega} - L\right)\vec{x}^{(k)} = \left[U + \frac{1-\omega}{\omega}D\right]\vec{x}^{(k-1)} + \vec{b}.$$

They can also be written as fixed point problems: $\boxed{\vec{x} = T\vec{x} + \vec{c}}$

I. Jacobi: $\vec{x}^{(k)} = T_j\vec{x}^{(k-1)} + \vec{c}_j$, $\quad \boxed{T_j \equiv D^{-1}(L+U), \quad \vec{c}_j = D^{-1}\vec{b}.}$

II. G-S.: $\vec{x}^{(k)} = T_{GS}\vec{x}^{(k-1)} + \vec{C}_{GS}$, $\quad \boxed{T_{GS} \equiv (D-L)^{-1}U, \quad C_{GS} = (D-L)^{-1}\vec{b}}$

III. SOR: $\vec{x}^{(k)} = T_\omega\vec{x}^{(k-1)} + \vec{C}_\omega$, $\quad \boxed{T_\omega \equiv (D-\omega L)^{-1}[\omega U + (1-\omega)D], \quad \vec{C}_\omega \equiv \omega(D-\omega L)^{-1}\vec{b}.}$

# Summary 2

$$A\vec{x} = \vec{b}$$

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1i} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2i} & \cdots & a_{2n} \\ \vdots & & \ddots & & & \\ a_{i1} & & & a_{ii} & \cdots & a_{in} \\ & & & & \ddots & \\ a_{n1} & \cdots & & & & a_{nn} \end{pmatrix}$$

$$A = \underbrace{\begin{pmatrix} a_{11} & & & \\ & \ddots & & \\ & & a_{ii} & \\ & & & \ddots \\ & & & & a_{nn} \end{pmatrix}}_{} + \underbrace{\begin{pmatrix} 0 & & & 0 \\ a_{21} & 0 & & \\ \vdots & & \ddots & \\ a_{n1} & & & 0 \end{pmatrix}}_{= -L} + \underbrace{\begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ & \ddots & & \vdots \\ 0 & & \ddots & a_{n-1,n} \\ & & & 0 \end{pmatrix}}_{= -U}$$

or $\quad A = D$

or $\quad A\vec{x} = \vec{b} \iff D\vec{x} = (L+U)\vec{x} + \vec{b}$

or $\quad A\vec{x} = \vec{b} \iff (D-L)\vec{x} = U\vec{x} + \vec{b}$

## For Computation

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}\, x_j^{(k-1)} + b\right]$$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k-1)} + b_i\right]$$

$$x_i^{(k)} = \frac{\omega}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}\, x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}\, x_j^{(k-1)} + b_i\right] + (1-\omega)\, x_i^{(k-1)}$$

## For Analysis.

$$\vec{x} = D^{-1}(L+U)\vec{x} + D^{-1}\vec{b}, \quad a_{ii} \neq 0$$

or $\boxed{\vec{x} = T_j\,\vec{x} + \vec{c}_j}$

$$(D-L)\vec{x} = U\vec{x} + \vec{b}, \quad \text{if } D-L \text{ invertible}$$

$$\boxed{\vec{x} = (D-L)^{-1}U\,\vec{x} + (D-L)^{-1}\vec{b}}$$

or $\vec{x} = T_{GS}\,\vec{x} + \vec{c}_{GS}$

$$(D-\omega L)\,\vec{x} = \omega U\vec{x} + (1-\omega)D\,\vec{x} + \omega I\vec{b}$$

If $D-\omega L$ is invertible

$$\boxed{\vec{x} = \underbrace{(D-\omega L)^{-1}\left[\omega U + (1-\omega)D\right]}_{T_\omega}\vec{x} + \underbrace{\omega(D-L\omega)^{-1}\vec{b}}_{\vec{c}_\omega}}$$

or $\vec{x} = T_\omega\,\vec{x} + \vec{c}_\omega$

The condition $0 < \omega < 2$ is also sufficient for special set of matrices $A$.

Thm. (Ostrowski- Reich).
If $A$ is positive definite and $\omega \in (0,2)$ then, SOR converges for any choice of initial approximation $\vec{v}^{(0)}$.

Recall $A$ is positive definite if $\vec{x}^T A \vec{x} > 0$ for any $\vec{x} \neq \vec{0}$, and $A$ is Symmetric.

Also, $A$ is P.D $\Leftrightarrow$ All eigenvalues of $A$ are positive.

Thm. If $A$ is P.D. and tridiagonal, then $\rho(M_g) = \rho^2(M_j) < 1$ and the optimal choice of $\omega$ for SOR method is

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho^2(M_j)}}$$

and $\rho(M_\omega) = \omega - 1$.

Notice $1 < \omega_{opt} < 2$

## Application of previous theorem to our linear system.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

A is P.D since $|2| = 2 > 0$, $\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5 > 0$ and $|A| = 12 - 4 = 8 > 0$. and it's also tridiag. then the prev. Thm. can be applied to A. A is Symm.

$$A = D - L - U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \Rightarrow D\vec{x} = (L+U)\vec{x} + \vec{b} \Rightarrow \vec{x} = D^{-1}(L+U)\vec{x} + D^{-1}\vec{b}$$

or $\boxed{\vec{x} = M_j \vec{x} + \vec{C}_j}$, $M_j \equiv D^{-1}(L+U)$.

For our example,

$$M_j = \underbrace{\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}_{D^{-1}} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{(L+U)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix}$$

$$\Rightarrow |M_j - \lambda I| = \begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/3 & -\lambda & 1/3 \\ 0 & 1/2 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{2}{6}\lambda = 0 \Rightarrow \lambda\left(\lambda^2 - \frac{2}{6}\right) = 0$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_{2,3} = \pm \frac{\sqrt{2}}{\sqrt{2}\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

$$\Rightarrow \rho(M_j) = \frac{\sqrt{3}}{3} \simeq 0.58.$$

$\Rightarrow$ Optimal $\omega$, $\quad \omega_{opt} = \dfrac{2}{1 + \sqrt{1 - \rho^2(M_j)}} \simeq \underline{\underline{1.1.}}$ ✓

# Analysis of Convergence

Consider the fixed point problem:

$$\vec{x} = T\vec{x} + \vec{c}, \quad \vec{x} \in \mathbb{R}^n \qquad (5.1)$$

and assume, it has a unique fixed point $\vec{x}^*$.
It means that there is a $\vec{x}^*$ that satisfies the equation (5.1),

$$\vec{x}^* = T\vec{x}^* + \vec{c}. \qquad (5.2)$$

Also, consider the iterative technique corresponding to (5.1) given by

$$\vec{x}^{(k)} = T\vec{x}^{(k-1)} + \vec{c} \qquad (5.3)$$

Theorem:

The iteration (5.3) converges to the fixed point $\vec{x}^*$ of (5.1) if $\|T\| < 1$

Proof. Next page

Subtracting (5.2) from (5.3) and using linearity leads to

$$\vec{x}^{(k)} - \vec{x}^* = T(\vec{x}^{(k-1)} - \vec{x}^*) \qquad (5.4)$$

defining $\vec{e}^{k} = \vec{x}^{(k)} - \vec{x}^*$

Equ. (5.4) reduces to

$$\vec{e}^{(k)} = T\,\vec{e}^{(k-1)}$$

Iterating the right term

$$\vec{e}^{(k)} = T\vec{e}^{(k-1)} = T(T\vec{e}^{k-2}) = T^2 e^{(k-2)} = \dots$$
$$= T^k \vec{e}^{(0)}$$

Thus, Using any norm in $\mathbb{R}^n$

$$\|\vec{e}^{(k)}\| = \|T^k\vec{e}^{(0)}\| \le \|T^k\|\,\|\vec{e}^{(0)}\| \le \|T\|^k\,\|\vec{e}^{(0)}\|$$

Because $\|T\| < 1 \Rightarrow \|T\|^k \xrightarrow[k\to\infty]{} 0$

and $\|\vec{e}^{(k)}\| \le \|T^k\|\,\|\vec{e}^{(0)}\| \xrightarrow[k\to\infty]{} 0$

or $\|x^{(k)} - \vec{x}^*\| \xrightarrow[k\to\infty]{} 0$ ✓