6.5 Iterative Solution of Linear equations.

(Burden-Faires)

$$\begin{cases} 2x_1 - x_2 = 1 \\ -x_1 + 3x_2 - x_3 = 8 \\ \quad -x_2 + 2x_3 = -5 \end{cases} \iff A\vec{x} = \vec{b} \qquad (1.1)$$

$$\underbrace{\begin{pmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{pmatrix}}_{=A} \underbrace{\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}}_{=\vec{x}} = \underbrace{\begin{pmatrix} 1 \\ 8 \\ -5 \end{pmatrix}}_{=\vec{b}} \iff A\vec{x} = \vec{b}$$

$$\begin{cases} x_1 = \frac{1}{2}x_2 + \frac{1}{2} \\ x_2 = \frac{1}{3}x_1 + \frac{1}{3}x_3 + \frac{8}{3} \\ x_3 = \frac{1}{2}x_2 - \frac{5}{2} \end{cases} \qquad (1.2)$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{3} & 0 & \frac{1}{3} \\ 0 & \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} + \begin{pmatrix} 1/2 \\ 8/3 \\ -5/2 \end{pmatrix}$$

$$\vec{x} = T\vec{x} + \vec{c} \qquad (1.3)$$

We are replacing $A\vec{x} = \vec{b}$ by the equivalent System $\vec{x} = T\vec{x} + \vec{c}$

Jacobi Iteration:

Start with an arbitrary initial approximation

$$\vec{x}^{(0)} = \begin{pmatrix} x_1^{(0)} \\ x_2^{(0)} \\ x_3^{(0)} \end{pmatrix}$$

and follow with the iterative formula:

$$\vec{x}^{(k)} = T\,\vec{x}^{(k-1)} + \vec{c}$$

$$\begin{cases} x_1^{(k)} = \frac{1}{2} x_2^{(k-1)} + \frac{1}{2} \\ x_2^{(k)} = \frac{1}{3} x_1^{(k-1)} + \frac{1}{3} x_3^{(k-1)} + \frac{8}{3} \\ x_3^{(k)} = \frac{1}{2} x_2^{(k-1)} - \frac{5}{2} \end{cases}$$

Starting with $\vec{x}^{(0)} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$

we find $\vec{x}^{(1)} = (0.5,\ 2.66,\ -2.5)^T$

$\vdots$

$\vec{x}^{(21)} = (2, 3, -1)^T$ exact soln.

In general,

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1i}x_i + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2i}x_i + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{ii}x_i + \cdots + a_{in}x_n = b_i \\
\vdots \\
a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{ni}x_i + \cdots + a_{nn}x_n = b_n
\end{cases}
$$

is equivalent to

$$a_{11}x_1 = -a_{12}x_2 - \cdots - a_{1i}x_i - \cdots - a_{1n}x_n + b_1$$

$$a_{22}x_2 = -a_{21}x_1 - a_{23}x_3 - \cdots - a_{2i}x_i \cdots - a_{2n}x_n + b_2$$

$$\vdots$$

$$a_{ii}x_i = -a_{i1}x_1 - \cdots - a_{i,i-1}x_{i-1} - a_{i,i+1}x_{i+1} - \cdots - a_{in}x_n + b_i$$

$$\vdots$$

$$a_{nn}x_n = -a_{n1}x_1 - \cdots - a_{n,i}x_i - \cdots - a_{n,n-1}x_{n-1} + b_n$$

or in more compact form

$$a_{ii}x_i = -\sum_{\substack{j=1 \\ j\neq i}}^{n} a_{ij}x_j + b_i, \qquad i = 1, .., n. \quad (3.1)$$

or $D\vec{x} = (L+U)\vec{x} + \vec{b}.$

If $a_{ii} \neq 0$

$$x_i = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}x_j + b_j\right] \qquad (4.1)$$

$$i=1,..,n$$

or in matrix form:

$$\begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_i \\ \vdots \\ x_n \end{pmatrix} = -\begin{pmatrix} 0 & \frac{a_{12}}{a_{11}} & \frac{a_{13}}{a_{11}} & \cdots & \frac{a_{1i}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ \frac{a_{21}}{a_{22}} & 0 & \frac{a_{23}}{a_{22}} & \cdots & \frac{a_{2i}}{a_{22}} & \cdots & \frac{a_{2n}}{a_{22}} \\ \vdots & & & & & & \\ \frac{a_{i1}}{a_{ii}} & \frac{a_{i2}}{a_{ii}} & \cdots & \frac{a_{i,i-1}}{a_{ii}} & 0 & \frac{a_{i,i+1}}{a_{ii}} \cdots & \frac{a_{in}}{a_{ii}} \\ \vdots & & & & & & \\ \frac{a_{n1}}{a_{nn}} & \frac{a_{n2}}{a_{nn}} & \cdots & & \frac{a_{ni}}{a_{nn}} & \cdots \frac{a_{n,n-1}}{a_{nn}} & 0 \end{pmatrix} \vec{x} + \begin{pmatrix} b_1/a_{11} \\ b_2/a_{22} \\ \vdots \\ b_i/a_{ii} \\ \vdots \\ b_n/a_{nn} \end{pmatrix}$$

or $\quad \vec{x} = T\vec{x} + \vec{c} \quad$ or $\quad \vec{x} = D^{-1}(L+U)\vec{x} + D^{-1}\vec{b} \qquad (4.2)$

Jacobi Iteration:

$$\vec{x}^{(k)} = T\,\vec{x}^{(k-1)} + \vec{c} \qquad (5.1)$$

$$x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{\substack{j=1\\ j\neq i}}^{n} a_{ij}\,x_j^{(k-1)} + b_i\right] \qquad (5.2)$$

$i = 1, .., n$

Gauss-Siedel iteration

In our previous example,

$$\begin{cases} x_1^{(k)} = \frac{1}{2}\,x_2^{(k-1)} + \frac{1}{2} \\[2mm] x_2^{(k)} = \frac{1}{3}\,x_1^{(k)} + \frac{1}{3}\,x_3^{(k-1)} + \frac{8}{3} \\[2mm] x_3^{(k)} = \frac{1}{2}\,x_2^{(k)} - \frac{5}{2} \end{cases} \qquad (5.3)$$

Starting with $\vec{x}^{(0)} = (0,0,0)$.

$$\vec{x}^{(1)} = (0.5,\ 2.8,\ -1.08)^T$$

$$\vec{x}^{(2)} = (1.9,\ 2.9,\ -1.02)^T$$

$$\vdots$$

$$\vec{x}^{(9)} = (2,\ 3,\ -1)^T$$

In General for any linear system, $A\vec{x}=\vec{b}$ Gauss-Seidel Algorithm Can be written as (A nonsingular)

$$\begin{cases} a_{11}x_1^{(k)} = -a_{12}x_2^{(k-1)} - a_{13}x_3^{(k-1)} - \dots - a_{1n}x_n^{(k-1)} + b_1 \\ a_{21}x_1^{(k)} + a_{22}x_2^{(k)} = -a_{23}x_3^{(k-1)} - \dots - a_{2n}x_n^{(k-1)} + b_2 \\ \vdots \\ a_{i1}x_1^{(k)} + a_{i2}x_2^{(k)} + \dots + a_{ii}x_i^{(k)} = -a_{i,i+1}x_{i+1}^{(k-1)} + \dots + a_{i,n}x_n^{(k-1)} + b_i \\ \vdots \\ a_{n1}x_1^{(k)} + a_{n2}x_2^{(k)} + \dots + a_{nn}x_n^{(k)} = \dots + b_n \end{cases}$$

$$\boxed{x_i^{(k)} = \frac{1}{a_{ii}}\left[-\sum_{j=1}^{i-1} a_{ij}x_j^{(k)} - \sum_{j=i+1}^{n} a_{ij}x_j^{(k-1)}\right] + \frac{b_i}{a_{ii}}, \quad i=1,2,..,n}$$

↳ Formula used for computation.

Theoretical formula:

$$\boxed{(D-L)\,\vec{x}^{(k)} = U\vec{x}^{(k-1)} + \vec{b}.}$$

$$\boxed{\begin{aligned} \vec{x}^{(k)} &= (D-L)^{-1}U\,\vec{x}^{(k-1)} + (D-L)^{-1}\vec{b} \\ \text{or} \quad \vec{x}^{(k)} &= T_g\vec{x}^{(k-1)} + \vec{C}_g \end{aligned}}$$

Summarizing Gauss-Seidel.

$$a_{ii} x_i^{(k)} = - \sum_{\substack{j=1 \\ j<i}}^{n} a_{ij} x_j^{(k)} - \sum_{\substack{j=1 \\ j>i}}^{n} a_{ij} x_j^{(k-1)} + b_i \quad (6.1)$$

or

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[- \sum_{\substack{j=1 \\ j<i}}^{n} a_{ij} x_j^{(k)} - \sum_{\substack{j=1 \\ j>i}}^{n} a_{ij} x_j^{(k-1)} + b_i \right] \quad (6.2).$$

$i = 1, \ldots, n$

Succesive Overrelaxation (SOR)

Once $\vec{x}_i^{(k)}$ has been obtained from Gauss-Siedel iteration. The convergence can be accelerated computing a new $x_i^{(k)}$ by means

$$x_i^{(k)} = \omega x_i^{(k)} + (1-\omega) x_i^{(k-1)} \qquad (6.3)$$

Where $0 < \omega < 2$

Applying (6.3) to our example. It means
Combining (6.3) and (5.3).

$$
\begin{cases}
x_1^{(k)} = \omega\left[\frac{1}{2} x_2^{(k-1)} + \frac{1}{2}\right] + (1-\omega)\, x_1^{(k-1)} \\[2ex]
x_2^{(k)} = \omega\left[\frac{1}{3} x_1^{(k)} + \frac{1}{3} x_3^{(k-1)} + \frac{8}{3}\right] + (1-\omega)\, x_2^{(k-1)} \\[2ex]
x_3^{(k)} = \omega\left[\frac{1}{2} x_2^{(k)} - \frac{5}{2}\right] + (1-\omega)\, x_3^{(k-1)}
\end{cases}
$$

In general,

$$
x_i^{(k)} = \frac{\omega}{a_{ii}}\left[-\sum_{\substack{j=1\\ j<i}}^{n} a_{ij}\, x_j^{(k)} - \sum_{\substack{j=1\\ j>i}}^{n} a_{ij}\, x_j^{(k-1)} + b_i\right] + (1-\omega)\, x_i^{(k-1)} \qquad (7.1)
$$

$$i = 1, \ldots, n.$$

starting with $x^{(0)} = (0,0,0)^T$ in our example and $\boxed{\omega = 1.1}$

$\vec{x}^{(1)} = (0.55,\ 3.135,\ -1.026)^T$

$\vdots$

$\vec{x}^{(7)} = (2,\ 3,\ -1)$ exact soln.