CHAPTER 7

ITERATIVE TECHNIQUES IN LINEAR ALGEBRA.

Def.- Vector norm is a function

$$\|\cdot\| : \mathbb{R}^n \longrightarrow \mathbb{R}$$

with the following properties:

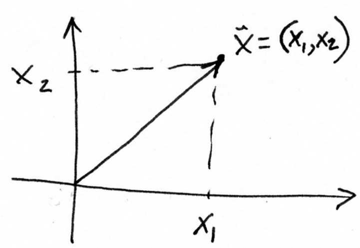
i) $\|\vec{x}\| \geq 0, \quad \forall \vec{x} \in \mathbb{R}^n$.

ii) $\|\vec{x}\| = 0 \Leftrightarrow \vec{x} = \vec{0}$.

iii) $\|\alpha \vec{x}\| = |\alpha| \, \|\vec{x}\|, \quad \forall \alpha \in \mathbb{R}$ and $\vec{x} \in \mathbb{R}^n$.

iv) $\|\vec{x} + \vec{y}\| \leq \|\vec{x}\| + \|\vec{y}\|, \quad \forall \vec{x}, \vec{y} \in \mathbb{R}^n$.

Example 1.- In $\mathbb{R}^2$ (plane), euclidean norm or distance.



$$\|\vec{x}\| = \sqrt{x_1^2 + x_2^2}$$

magnitude of the vector
distance from origin to the point $\vec{x}$.

Example 2.- The l_2 norm is defined in $\mathbb{R}^n$ as

$$\|\vec{x}\|_2 \equiv \left(\sum_{i=1}^{n} x_i^2 \right)^{1/2}$$

Example 3.- l_∞ norm of infinite norm

$$\|\vec{x}\|_\infty = \max_{1 \le i \le n} |x_i|$$

All these definitions are motivated for the necessity to measure errors. in the numerical approximations. The magnitude of the error will depend on the norm being used. In fact,

$\vec{x} = (2,3,5)$ is such that

$$\|\vec{x}\|_2 = \sqrt{4+9+25} = \sqrt{38}, \quad \text{while}$$

$$\|\vec{x}\|_\infty = \max\left(|2|,|3|,|5|\right) = 5.$$

An important theorem. Cauchy-Schwarz inequality

$$(\vec{x},\vec{y}) \le \|\vec{x}\|\,\|\vec{y}\|, \quad \text{or} \sum_{i=1}^{n} x_i y_i \le \|\vec{x}\|\,\|\vec{y}\|$$

Corollary.- For any $\vec{x}, \vec{y} \in \mathbb{R}^n$.

$$\|\vec{x}+\vec{y}\|_2 \le \|\vec{x}\|_2 + \|\vec{y}\|_2$$

Proof.- $$\|\vec{x}+\vec{y}\|_2^2 = \sum_{i=1}^{n} (x_i+y_i)^2 = \sum_{i=1}^{n} x_i^2 + 2\sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2$$

$$\le \|\vec{x}\|_2^2 + 2\|\vec{x}\|_2\|\vec{y}\|_2 + \|\vec{y}\|_2 \le \left(\|\vec{x}\|_2 + \|\vec{y}\|_2\right)^2$$

$\Longrightarrow$ TRIANGULAR INEQUALITY : $\|\vec{x}+\vec{y}\| \le \|\vec{x}\| + \|\vec{y}\|$.

<u>Definition</u>:- Distance between vectors in $\mathbb{R}^n$.

$$\vec{x} = (x_1, .., x_n), \qquad \vec{y} = (y_1, \dots y_n).$$

$$\|\vec{x}-\vec{y}\|_2 = \left[\sum_{i=1}^{n} (x_i - y_i)^2 \right]^{1/2} \text{ and } \|\vec{x}-\vec{y}\|_\infty = \max_{1 \le i \le n} |x_i - y_i|$$

<u>Theorem</u>.- For each $\vec{x} \in \mathbb{R}^n$.

$$\|\vec{x}\|_\infty \le \|\vec{x}\|_2 \le \sqrt{n}\, \|\vec{x}\|_\infty .$$

<u>Proof</u>.-

$$\|\vec{x}\|_\infty^2 = x_j^2 \le \sum_{i=1}^{n} x_i^2 = \|\vec{x}\|_2^2 \Rightarrow \boxed{\|\vec{x}\|_\infty^2 \le \|\vec{x}\|_2^2 .} \quad (3.1)$$

where $x_j = \max_{1 \le i \le n} |x_i| = \|\vec{x}\|_\infty$.

Also,

$$\|\vec{x}\|_2^2 = \sum_{i=1}^{n} x_i^2 \le \sum_{i=1}^{n} x_j^2 = n\, x_j^2 = n \max_{1 \le i \le n} |x_i|^2$$

$$= n \|\vec{x}\|_\infty^2 .$$

$$\Rightarrow \boxed{\|\vec{x}\|_2 \le \sqrt{n}\, \|\vec{x}\|_\infty} \qquad (3.2)$$

Combining (3.1) and (3.2).

$$\boxed{\|\vec{x}\|_\infty \le \|\vec{x}\|_2^2 \le \sqrt{n}\, \|\vec{x}\|_\infty}$$ ✓

Definition:- Consider the set of all matrices

$$\{A : A \text{ } n\times n \text{ matrix}\} = M_{n\times n}$$

The function

$$\|\cdot\| : M_{n\times n} \longrightarrow \mathbb{R}$$

satisfying the following properties for any matrices $A, B \in M_{n\times n}$ and $\alpha \in \mathbb{R}$ is called a matrix norm

i) $\|A\| \geq 0$

ii) $\|A\| = 0 \Leftrightarrow A = [0]$. matrix with all entries equal to zero.

iii) $\|\alpha A\| \leq |\alpha| \, \|A\|$

iv) $\|A+B\| \leq \|A\| + \|B\|$

v) $\|AB\| \leq \|A\| \, \|B\|$.

Thm.- (Natural or induced matrix norm)

If $\|\cdot\|$ is an arbitrary norm on $\mathbb{R}^n$, then

$$\|A\| \equiv \max_{\|\underline{x}\|=1} \|A(\underline{x})\|$$

is a matrix norm.

Proof.- It's easy.

An alternative representation is given by choosing $\vec{z} \neq \vec{0}$ any vector and considering

$\vec{x} = \vec{z}/\|\vec{z}\|$ unit vector. All unit vectors can be obtained from some $\vec{z} \neq 0$

Then

$$\max_{\|\vec{x}\|=1} |A\vec{x}| = \max_{\|\vec{z}\| \neq \vec{0}} \left\| A\left(\frac{\vec{z}}{\|\vec{z}\|}\right)\right\| =$$

$$= \max_{\vec{z} \neq \vec{0}} \frac{\|A\vec{z}\|}{\|\vec{z}\|}$$

Thus,
$$\boxed{\|A\| = \max_{\vec{z} \neq \vec{0}} \frac{\|A\vec{z}\|}{\|\vec{z}\|}} \qquad (5.1)$$

Corollary.– For any $\vec{z} \neq \vec{0}$ and any natural norm $\|\cdot\|$

$$\|A\vec{z}\| \leq \|A\| \, \|\vec{z}\|.$$

Proof.– For any $\vec{z} \neq \vec{0}$, (5.1) implies

$$\frac{\|A\vec{z}\|}{\|\vec{z}\|} \leq \max \frac{\|A\vec{z}\|}{\|\vec{z}\|} = \|A\|$$

$$\Rightarrow \|A\vec{z}\| \leq \|A\| \, \|\vec{z}\|.$$

For our norms l_2 and l_∞ in $\mathbb{R}^n$,
we have two natural matrix norms. They are

$$\|A\|_\infty \equiv \max_{\|x\|_\infty = 1} \|A\vec{x}\|_\infty$$

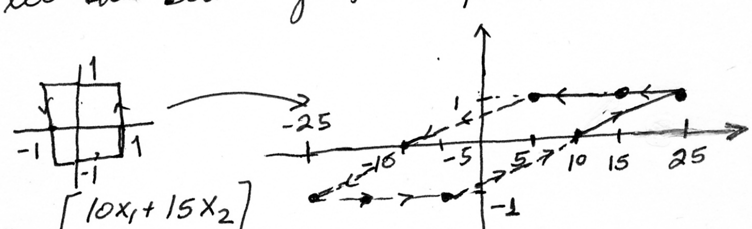
$$\|A\|_2 \equiv \max_{\|x\|_2 = 1} \|A\vec{x}\|_2$$

Therefore, the norm of a matrix (as we have defined it) is the maximum stretch of the unit vectors.

Exercise 4 a) Find $\| \ \|_\infty$ of

$A = \begin{bmatrix} 10 & 15 \\ 0 & 1 \end{bmatrix}$ if $\vec{x}$ is such that $\|\vec{x}\|_\infty = 1$,

then $\vec{x}$ lies in the boundary of a square of sides 1 in the plane.



$$A\vec{x} = \begin{bmatrix} 10 & 15 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 10x_1 + 15x_2 \\ x_2 \end{bmatrix}$$

$$\Rightarrow \|A\vec{x}\|_\infty = \max\left(|10x_1 + 15x_2|, |x_2| \right)$$

Obviously, for vectors lying in the square the maximum is reached when $x_1 = 1$, $x_2 = 1$, for the vectors $\vec{x} = (1,1)$.

Thus $\quad \|A\|_\infty = \max_{\|\vec{x}\| = 1} \|A\vec{x}\|_\infty = |10x_1 + 15x_2| = \underline{\underline{25}}$

Thm.- $A = (a_{ij})$ is an $n \times n$ matrix, then

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

Proof.- Let's show that first

$$\|A\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|$$

Let $\hat{x}$ be such that $\|\hat{x}\|_\infty = 1$

$$\Rightarrow \|A\hat{x}\|_\infty = \max_{1 \le i \le n} |(A\hat{x})_i| = \max_{1 \le i \le n} \left| \sum_{j=1}^{n} a_{ij} x_j \right| \le$$

$$\overset{\text{Triang. ineq.}}{\le} \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}||x_j| \le$$

$$\le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \left(\max_{1 \le j \le n} |x_j| \right) = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}| \overbrace{\|\hat{x}\|_\infty}^{=1}$$

$$\therefore \quad \boxed{\|A\|_\infty \le \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|} \qquad (7.1)$$