4.2 Neumann Boundary Conditions.

$$\begin{cases} u_t = \sigma u_{xx}, & 0 < x < 1, \quad t > 0, \\ \text{I.C.} \quad u(x,0) = \phi(x), & 0 \le x \le 1 \\ \text{B.C's} \begin{cases} u(0,t) = f(t), \\ u_x(1,t) = g(t), \end{cases} & t > 0 \end{cases}$$

$u_x(1,t) = g(t)$ $\downarrow$ Neumann condition.

Let's use the weighted average scheme

$$\boxed{\begin{aligned} -r\theta v_{j-1}^{n+1} + (1+2r\theta) v_j^{n+1} - r\theta v_{j+1}^{n+1} &= \\ = r(1-\theta) v_{j-1}^n + [1-2r(1-\theta)] v_j^n &+ r(1-\theta) v_{j+1}^n \end{aligned}} \qquad (1.1)$$

$\theta \in [0,1]$, $r = \sigma \Delta t / \Delta x^2$.

(I) We already studied the case of periodic initial and boundary conditions over the interval $[0, 2\pi]$ and we were able to prove

a) Scheme (1.1) is unconditionally stable in L_2 norm when $1/2 \le \theta \le 1$, i.e., $\|\vec{v}^n\|_2 \le \|\vec{v}^0\|_2$, for all r.

b) Scheme (1.1) is absolutely stable, i.e., $\|\vec{v}^n\|_2 \le \|\vec{v}^0\|_2$ if $r \le 1/2(1-2\theta)$, when $0 \le \theta < 1/2$.

(II) We also obtained a matrix representation for this scheme for homogeneous Dirichlet Condition

(III) Now, if there is a Neumann condition at $x=1$:

$$\frac{\partial u}{\partial x}(1,t) = g(t)$$

and Dirichlet condition at $x=0$:

$$u(0,t) = f(t).$$

and the I.C. $u(x,0) = \phi(x)$, $0 \le x \le 1$.

Then $v_j^0 = \phi(x_j)$, $j = 0, 1, .., J$.

For the subsequent level in times $t_n = n * \Delta t$, $n = 1, 2 \dots$

Equ. (1.1). is still the same for $j = 1, 2.., J-1$

But now, for $j = J-1$

new unkwn

$j = J-1$

$$-r\theta v_{J-2}^{n+1} + (1+2r\theta) v_{J-1}^{n+1} - r\theta v_J^{n+1} = \qquad (2.1)$$

$$r(1-\theta) v_{J-2}^{n} + [1-2r(1-\theta)] v_{J-1}^{n} + r(1-\theta) v_J^{n}$$

The term v_J^n is know from the previous time step and v_J^{n+1} is a new unknown. Since, we have $J-1$ equs. and J unknowns one more equation is needed.

Alternatives:

a) Approximate (Backward 2 points)

$$(U_x)_J^{n+1} \simeq \frac{U_J^{n+1} - U_{J-1}^{n+1}}{\Delta x} = g^{n+1}$$

$$\Rightarrow U_J^{n+1} = U_{J-1}^{n+1} + \Delta x\, g^{n+1}$$ New equ.

Disadvantage: Approx. $\mathcal{O}(\Delta x)$ lower order than scheme.

b) Approximate (Backward 3 points)

$$(U_x)_J^{n+1} = \frac{U_{J-2}^{n+1} - 4\,U_{J-1}^{n+1} + 3\,U_J^{n+1}}{2\Delta x} + \mathcal{O}(\Delta x^2).$$

O.K. approx. of U_x Same order as scheme.

New equ.

$$U_J^{n+1} = \frac{1}{3}\left[4\,U_{J-1}^{n+1} - U_{J-2}^{n+1} + 2\Delta x\, g^{n+1}\right]$$

c) Approximation (Centered 3 points)

$$(U_x)_J^{n+1} = \frac{U_{J+1}^{n+1} - U_{J-1}^{n+1}}{2\Delta x} + \mathcal{O}(\Delta x^2)$$

$\Rightarrow$ New Equ.

$$\boxed{U_{J+1}^{n+1} = U_{J-1}^{n+1} + 2\Delta x\, g^{n+1}} \qquad (3.1)$$

Disadvantage:

Add one more unknown U_{J+1}^{n+1} Value of discrete approximation at a fictitious point (x_{J+1}, t_{n+1}), also called Ghost Point.

Now, we have J equations and $J+1$ unknowns.

$$U_j^{n+1}, \quad j = 1, 2, \ldots, J, J+1.$$

We can add one more equation if we replace $j \to J$ in (2.1). In fact,

$$-r\theta\, U_{J-1}^{n+1} + (1+2r\theta) U_J^{n+1} - r\theta\, U_{J+1}^{n+1} =$$

New unknown

$$r(1-\theta) U_{J-1}^{n} + [1 - 2r(1-\theta)] U_J^{n} + r(1-\theta) U_{J+1}^{n}.$$

Another new unknown appears U_{J+1}^n. Now, we have $J+1$ equs. and $J+2$ unknowns. (4.1)

An additional equ. can be obtained from (3.1) again replacing $n+1 \longrightarrow n$

$$\boxed{U_{J+1}^{n} = U_{J-1}^{n} + 2\Delta x\, g^{n}} \qquad (4.2)$$

Summarizing:

System of equations for Neumann condition

$$\frac{\partial u}{\partial x}(L,t) = g(t), \quad \text{at} \quad x = L$$

and Dirichlet cond. at $x=0$

$$u(0,t) = f(t).$$

$j=1$:

$$(1+2r\theta)U_1^{n+1} - r\theta U_2^{n+1} = r(1-\theta)\underbrace{U_0^n}_{=f^n} + [1-2r(1-\theta)]U_1^n + r(1-\theta)U_2^n + r\theta \boxed{U_0^{n+1} = f^{n+1}}$$

Known B.C.

$j = 2, .., J-1$

$$-r\theta U_{j-1}^{n+1} + (1+2r\theta)U_j^{n+1} - r\theta U_{j+1}^{n+1} = r(1-\theta)U_{j-1}^n + [1-2r(1-\theta)]U_j^n + r(1-\theta)U_{j+1}^n$$

$j=J$

New fictitious

$$-r\theta U_{J-1}^{n+1} + (1+2r\theta)U_J^{n+1} - r\theta U_{J+1}^{n+1} = r(1-\theta)U_{J-1}^n + [1-2r(1-\theta)]U_J^n + r(1-\theta)U_{J+1}^n$$

New fictitious

Two more equation from approx. of Neumann B.C.

$$U_{J+1}^n = U_{J-1}^n + 2\Delta x \underbrace{(u_x)_J^n}_{=g^n}$$

$$U_{J+1}^{n+1} = U_{J-1}^{n+1} + 2\Delta x \underbrace{(u_x)_J^{n+1}}_{=g^{n+1}}$$

Centered Finite DIFFCE Approx.

J+2 equations and J+2 unkws.

The last 3 equations are combined in one and the values U_{J+1}^{n} and U_{J+1}^{n+1} at fictitious points are eliminated. So,

$$\boxed{j = J: \; -2r\theta\, U_{J-1}^{n+1} + (1+2r\theta)\, U_{J}^{n+1} = 2r(1-\theta)\, U_{J-1}^{n} + \left[1-2r(1-\theta)\right] U_{J}^{n} + 2r(1-\theta)\Delta x\, g^{n} + 2r\theta\Delta x\, g^{n+1}} \quad (5.1)$$

Now, we have J equations with $U_1^{n+1}, \dots, U_J^{n+1}$ J unknowns.

Assuming $f(t) = u(0,t) = 0$ and $g(t) = u_x(1,t) = 0$

The linear system to be solved reduces to

$$\boxed{\left[I - r\theta C\right] \vec{U}^{n+1} = \left[I + r(1-\theta)C\right] \vec{U}^{n},} \quad (5.2)$$

where

$$\vec{U}^{n} = \begin{bmatrix} U_1^n \\ \vdots \\ U_J^n \end{bmatrix}, \qquad C = \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 2 & -2 \end{bmatrix}_{J\times J}.$$

Also (5.2) can also be written as

$$\boxed{\vec{U}^{n+1} = \left[I - r\theta C\right]^{-1}\left[I + r(1-\theta)C\right]\vec{U}^{n} = L_\Delta \vec{U}^{n}} \quad (6.1)$$

We want to study the stability of this numerical scheme using matrix analysis.

Useful Linear Algebra results:

Theorem.

If A, B $(n \times n)$ matrices are similar, then they have same characteristic polynomials and hence same eigenvalues.

Proof:

If $B = P^{-1}AP$

$$\Rightarrow B - \lambda I = P^{-1}AP - \lambda P^{-1}P = P^{-1}(AP - \lambda P) = P^{-1}(A - \lambda I)P$$

$$\Rightarrow \det(B - \lambda I) = \det(P^{-1})\det(A - \lambda I)\det(P)$$

$$= \det(A - \lambda I)\det(P^{-1})\det(P)$$

$$= \det(A - \lambda I)\det(P^{-1}P) = \det(A - \lambda I).$$

Theorem.

If L_Δ is similar to a symmetric matrix Q

$$L_\Delta = P^{-1} Q P.$$

and $\|P\|$ and $\|P^{-1}\|$ are uniformly bounded.

The condition (Von Neumann condition)

$$\rho(L_\Delta) \le 1 + \hat{C}\Delta t, \qquad \begin{array}{l} \Delta t \le \Delta t^* \\ \Delta x \le \Delta x^* \\ n\Delta t \le T \end{array}$$

and $\hat{C}$ independent of $\Delta t, \Delta x$

is sufficient for stability in the L_2-norm.

Proof: If $L_\Delta = P^{-1} Q P$, Q symm

Notice

$$L_\Delta^2 = (P^{-1}QP)(P^{-1}QP) = (P^{-1}Q)(PP^{-1})$$

$$= P^{-1}Q^2P$$

In general,

$$L_\Delta^n = P^{-1} Q^n P.$$

Then,

$$\|L_\Delta^n\|_2 \le \|P^{-1}Q^nP\| \le \|P^{-1}\|_2\|Q^n\|_2\|P\|_2 \le \|P^{-1}\|_2\|Q\|_2^n\|P\|_2$$

$$\overset{\text{Symm } Q}{=\!=} \|P^{-1}\|\,\rho^n(Q)\,\|P\| \overset{\text{Prev. Thm}}{=\!=} \|P^{-1}\|\,\rho^n(L_\Delta)\,\|P\|$$

$$\le K\,\rho^n(L_\Delta) \overset{\text{Hip.}}{\le} K\,(1+c\Delta t)^n \le K\,e^{cn\Delta t}$$

$$\le K\,e^{cT} = \hat{C}$$

Thus, $\|L_\Delta^n\|_2 \le \hat{C}$ and FDM is stable.

So our goal in what follows is to apply this theorem to show stability in the L_2-norm for the weighted average scheme.

Back to stability analysis of the weighted average Scheme

$$\vec{U}^{n+1} = L_\Delta \vec{U}^n$$

Where

$$L_\Delta = \left[I - r\theta C\right]^{-1}\left[I + r(1-\theta)C\right]$$

According to previous theorem, we need to prove:

① L_Δ is Similar to a Symmetric matrix.

② $\rho(L_\Delta) \le 1 + c\Delta t$, $\quad \Delta x \le \Delta x^*$
$\Delta t \le \Delta t^*$
$n \to \infty, \quad n\Delta t \le T.$

We will actually prove a stronger condition

$$\rho(L_\Delta) \le 1.$$

Proof of ① requires to find P, P^{-1}, and $\tilde{L}_\Delta$ Such that

$$L_\Delta = P^{-1}\tilde{L}_\Delta P \quad \text{and} \quad (\tilde{L}_\Delta)^T = \tilde{L}_\Delta.$$

or $\tilde{L}_\Delta = D^{-1} L_\Delta D$, $\quad$ where $D^{-1} = P$, $\quad D = P^{-1}$

Lemma 1

If D is invertible

$$\tilde{L}_\Delta = D^{-1} L_\Delta D = (I - r\theta\tilde{C})^{-1}(I + r(1-\theta)\tilde{C})$$

where $\tilde{C} = D^{-1} C D$.

Proof.

$$\tilde{L}_\Delta = D^{-1} L_\Delta D = \left(D^{-1}[I - r\theta C]^{-1} D\right)\left(D^{-1}(I + r(1-\theta)C)D\right)$$

$$= \left[D^{-1}[I - r\theta C]D\right]^{-1}\left(I + r(1-\theta)D^{-1}CD\right)$$

$$= \left(I - r\theta D^{-1}CD\right)^{-1}\left(I + r(1-\theta)D^{-1}CD\right)$$

If $\tilde{C} = D^{-1}CD$ then,

$$\tilde{L}_\Delta = [I - r\theta\tilde{C}]^{-1}[I + r(1-\theta)\tilde{C}]$$

Lemma 2. If $\tilde{C}$ is Symm. $(\tilde{C}^T = \tilde{C})$

then, $[I - r\theta\tilde{C}]$ and $[I + r(1-\theta)\tilde{C}]$ are also Symm.

Proof. $[I - r\theta\tilde{C}]^T = I - r\theta\tilde{C}^T = [I - r\theta\tilde{C}]$

Similarly $[I + r(1-\theta)\tilde{C}]^T = [I + r(1-\theta)\tilde{C}]$

Theorem. If $\tilde{C} = D^{-1}CD$ is Symmetric $\Rightarrow \tilde{L}_\Delta$ is symmetric.

Proof. Calling $A \equiv [I - r\theta\tilde{C}]$, $B \equiv [I + r(1-\theta)\tilde{C}]$

$$\tilde{L}_\Delta = A^{-1}B$$

Therefore,

$$\tilde{L}_\Delta^T = (A^{-1}B)^T = B^T(A^{-1})^T = B^T(A^T)^{-1} \overset{\text{Lemma 2}}{=} B\,A^{-1}$$

So for showing symm. is enough to prove that

$$A^{-1}B = BA^{-1} \quad \text{(commute)}.$$

We first notice that $AB = BA$
by direct product of matrices.

Now, $(BA)^{-1}(BA) = I \Longrightarrow (AB)^{-1}(BA) = I$

Thus, $B^{-1}(A^{-1}BA) = I \Rightarrow A^{-1}BA = B$.

then, $(A^{-1}BA)A^{-1} = BA^{-1}$

or $A^{-1}B = (A^{-1}B)(AA^{-1}) = BA^{-1}$ ✓

which is equivalent to

$$\tilde{L}_\Delta = \tilde{L}_\Delta^T$$ ✓

Similarly, if

C were symm. $\Rightarrow$ L_Δ would be symm.

but C is not symm. for Neumann boundary Condition.

However, according to our previous theorems and lemmas

If we can find D invertible such that

$$\tilde{C} = D^{-1} C D$$

is symmetric, then $\tilde{L}_\Delta = D^{-1} L_\Delta D$ is also symmetric.

<u>Corollary</u>. $\tilde{L}_\Delta = D^{-1} L_\Delta D$, where $D = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & \ddots & & \vdots \\ & & & & \sqrt{2} \end{bmatrix}$

is symmetric.

<u>Proof</u>. According to previous theorem, it is enoug to show that $\tilde{C} = D^{-1} C D$ is symm.

In fact,

$$\tilde{C} = D^{-1} C D = \begin{bmatrix} -2 & 1 & 0 & \cdots & \cdots & 0 \\ 1 & -2 & 1 & \cdots & \cdots & 0 \\ 0 & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & 1 & -2 & \ddots & \sqrt{2} \\ \vdots & & & 1 & \sqrt{2} & -2 \end{bmatrix}$$

is clearly symm.

$$\tilde{C}^T = \tilde{C}.$$

In fact,

$$\overset{CD}{\begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ & \ddots & \ddots & \ddots & & \\ & & & 1 & -2 & 1 \\ & & & & 2 & -2 \end{bmatrix} \begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & 0 & & 1 & \\ & & & & \sqrt{2} \end{bmatrix}}$$

$$= \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ & & \ddots & \ddots & \ddots & \\ & 0 & & 1 & -2 & \sqrt{2} \\ & & & & 2 & -2\sqrt{2} \end{bmatrix}$$

$D^{-1}(CD)$

$$\begin{bmatrix} 1 & & & & \\ & 1 & & & \\ & & \ddots & & \\ & & & 1 & \\ & & & & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} -2 & 1 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ & & \ddots & & & \vdots \\ & 0 & & 1 & -2 & \sqrt{2} \\ & & & & 2 & -2\sqrt{2} \end{bmatrix} = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 \\ 1 & -2 & 1 & & \\ & & \ddots & & \\ & & 1 & -2 & \sqrt{2} \\ & & 0 & \sqrt{2} & -2 \end{bmatrix}$$

To prove stability of

$$\hat{U}^{n+1} = L_\Delta \hat{U}^n$$

where $L_\Delta \equiv [I - r\theta]^{-1}[I + r(1-\theta)C]$

We still need to show that

$$\rho(L_\Delta) \leq 1 + C\Delta t$$

We can actually show

$$\rho(L_\Delta) \leq 1.$$

<u>Lemma 3.</u>

If λ is an eigenvalue of L_Δ, i.e., there is $\vec{z} \neq \vec{0}$

such that $L_\Delta \vec{z} = \lambda \vec{z}$, or $[L_\Delta - \lambda I]\vec{z} = \vec{0}$.

then

$$\lambda = 1 + \frac{\mu r}{1 - \mu r \theta}$$

where μ is an eigenvalue of C, i.e,

$$C\vec{v} = \mu \vec{v}, \qquad \vec{v} \neq \vec{0}.$$

<u>Proof.</u> If $L_\Delta z = \lambda \vec{z}$

then
$$[I - r\theta C]^{-1}[I + r(1-\theta)C]\vec{z} = \lambda \vec{z}$$

Thus,
$$[I + r(1-\theta)C]z = \lambda[I - r\theta C]\vec{z}$$

$$\Rightarrow \left([I + r(1-\theta)C] - \lambda[I - r\theta C]\right)\vec{z} = 0$$

or
$$\left([r(1-\theta) + \lambda\theta r]C - (\lambda-1)I\right)\vec{z} = 0$$

$$\Rightarrow \quad C\vec{z} = \left(\frac{\lambda - 1}{r(1-\theta) + \lambda r\theta}\right)\vec{z}$$

$\Rightarrow \quad \mu = \dfrac{\lambda - 1}{r(1-\theta) + \lambda r\theta}$ is an eigenvalue of C. (14.1)

Solving for λ, we will get the eigenvalues of L_Δ in terms of the eigenvalues of C.

From (14.1) $\quad \lambda - 1 = r(1-\theta)\mu + \lambda\mu r\theta$

$$\Rightarrow \lambda(1 - \mu r\theta) = 1 + r(1-\theta)\mu = \mu r + 1 - r\theta\mu.$$

$$\Rightarrow \boxed{\lambda = 1 + \frac{\mu r}{1 - \mu r\theta}} \quad ✓ \qquad (14.2)$$

Lemma 4.

The eigenvalues of G and corresponding eigenvectors $\vec{z}^k$ are

$$\mu_k = -4\sin^2\left(\frac{k\pi}{4J}\right), \quad k = 2m-1, \quad m = 0, 1, .., J-1$$

$$z^k_j = \sin\left(\frac{k\pi j}{2J}\right)$$

—— ∘ ——

Theorem.

The spectral radius

$$\rho(L_\Delta) \le 1,$$

1) if $\frac{1}{2} \le \theta \le 1$, for all values of $r = \frac{\sigma \Delta t}{\Delta x^2}$.

2) if $0 \le \theta < \frac{1}{2}$, for $r \le \frac{1}{2(1-2\theta)}$

Proof.

From the previous Lemmas 3, 4

$$\rho(L_\Delta) \le 1 \Leftrightarrow |\lambda_k| \le 1 \Leftrightarrow -1 \le 1 - \frac{4r\sin^2\left(\frac{k\pi}{4J}\right)}{1+4r\theta\sin^2\left(\frac{k\pi}{J}\right)} \le 1$$

$$\Leftrightarrow \quad \frac{4r\sin^2(k\pi/J)}{1+4r\theta\sin^2\left(\frac{k\pi}{J}\right)} \le 2$$

$$\Leftrightarrow 4r\sin^2(k\pi/J) \le 2 + 8r\theta\sin^2(k\pi/J) \quad /2$$

$$2r(1-2\theta)\sin^2(k\pi/J) \le 1. \qquad (16.1)$$

To satisfy (16.1) is sufficient

$$2r(1-2\theta) \le 1 \qquad (16.2)$$

For this to happen two cases arise:

1) If $\frac{1}{2} \le \theta \le 1 \Rightarrow 1-2\theta \le 0$.
$\Rightarrow$ (16.2) is verified for any r.

2) If $0 \le \theta < \frac{1}{2} \Rightarrow 1-2\theta > 0$
and (16.2) is verified only if

$$r \le \frac{1}{2(1-2\theta)}.$$

<u>Corollary</u>. The weighted average scheme (WAS) is

1) Unconditionally stable (no restriction in $\Delta t, \Delta x$)
if $\frac{1}{2} \le \theta \le 1$.

2) If $0 \le \theta < \frac{1}{2}$ then W.A.S. is stable
if

$$r = \frac{\sigma\Delta t}{\Delta x^2} \le \frac{1}{2(1-2\theta)}.$$