2.12 Neumann Boundary Conditions.

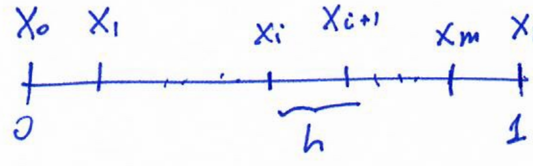
Consider BVP:

$$\begin{cases} u''(x) = f(x), & 0 < x < 1 \quad (1.1) \\ u'(0) = \sigma, \quad u(1) = \beta & \quad (1.2) \end{cases}$$

Discuss physical situation for heat conduction

Is a steady-state reached?

Discretization:



$x_{i+1} = x_i + h$, $i = 0, .., m$

Uniform partition $m+2$ points.

$$h = \frac{1}{m+1}, \quad v_i \simeq u(x_i)$$

Using centered difference approx. for (1.1) we obtain

$$\frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} = f(x_i), \quad i = 1, 2, .., m \qquad (1.3)$$

So, this is the equation (1.1) discretized at the interior points: $x_1, .., x_m$.

For each i there is an equation. For example

$$i = 1 \qquad \frac{v_2 - 2v_1 + v_0}{h^2} = f(x_1)$$

$$\vdots$$

$$i = m \qquad \frac{v_{m+1} - 2v_m + v_{m-1}}{h^2} = f(x_m)$$

Here, we only have m equations, but we also have $m+2$ unknowns: $v_0, v_1 .., v_m, v_{m+1}$

The linear System is completed by using the two boundary Conditions. $u(1)=\beta$ $\quad i=m+1: \; v_{m+1}=\beta$ or $\frac{1}{h^2}\left(h^2 v_{m+1}\right)=\beta$

We also need to discretize $u'(0)=\sigma$

Three cases will be study:

i) Forward difference approx. $\mathcal{O}(h)$ 1st order.

ii) Centered " approx. $\mathcal{O}(h^2)$ 2nd order.

iii) Second order one-sided approx. or right-sided difference in this case.

Case i) Forward difference: $(i=0)$

$$\sigma = u'(0) \simeq \frac{v_1 - v_0}{h}.$$

order of this approx.: $u'(0) = \frac{u(x_1)-u(x_0)}{h} + \mathcal{O}(h)$

$\Rightarrow$ Approx: $-\frac{1}{h}v_0 + \frac{1}{h}v_1 = \sigma$ or $\frac{1}{h^2}\left[-h v_0 + h v_1\right] = \sigma$

Then, linear system to be solved

$$\frac{1}{h^2}\begin{bmatrix} -h & h & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 \cdots & 0 \\ 0 & & \ddots & \ddots & \ddots & \vdots \\ \vdots & & & 1 & -2 & 1 \\ 0 & 0 & \cdots & & 0 & h^2 \end{bmatrix} \begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ \vdots \\ v_m \\ v_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma \\ f(x_1) \\ \vdots \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

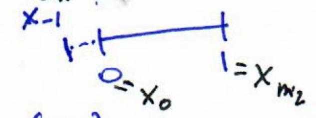
Comment on error (Global and Local). It is only $\mathcal{O}(h)$.

Case (ii): Centered difference approx of $u'(0)$.

$$u'(0) \simeq \frac{v_1 - v_{-1}}{2h}$$

v_{-1}: Value at fictitious point

NOTICE: $u'(0) = \dfrac{u(x_1) - u(x_{-1})}{2h} + \mathcal{O}(h^2)$



Approx: $\boxed{\dfrac{v_1 - v_{-1}}{2h} = \sigma}$

or $v_{-1} = -2h\sigma + v_1$ (3.1)

We need an extra equation for the unknown v_{-1}

That can be provided by the interior equation applied at $i=0$.

$$u''(0) \simeq \frac{v_{-1} - 2v_0 + v_1}{h^2}$$

Notice $u''(0) = \dfrac{u(x_{-1}) - 2u(x_0) + u(x_1)}{h^2} + \mathcal{O}(h^2)$.

Then the new discrete equation is

$$\frac{v_{-1} - 2v_0 + v_1}{h^2} = f(x_0) \quad (3.2)$$

Combining (3.1) and (3.2), we can eliminate v_{-1}. In fact,

$$\frac{v_1 - 2v_0 - 2h\sigma + v_1}{h^2} = f(x_0)$$

$$\Rightarrow \frac{2(v_1 - v_0)}{h^2} = f(x_0) + \frac{2\sigma}{h} \Rightarrow \boxed{\frac{1}{h^2}\left[-hv_0 + hv_1\right] = \sigma + \frac{h}{2} f(x_0)}$$

multiplying by $\frac{h}{2}$ both sides. to reobtain the same matrix A of forward diffce approx.

As a result, we obtain the linear system.

$$
\frac{1}{h^2}
\begin{bmatrix}
-h & h & 0 & 0 & 0 & \cdots & 0 \\
1 & -2 & 1 & 0 & 0 & \cdots & 0 \\
0 & 1 & -2 & 1 & 0 & \cdots & 0 \\
\vdots & & \ddots & \ddots & \ddots & & \vdots \\
 & & & 1 & -2 & 1 & \\
0 & & & & 0 & & h^2
\end{bmatrix}
\begin{bmatrix}
v_0 \\ v_1 \\ \vdots \\ \vdots \\ v_m \\ v_{m+1}
\end{bmatrix}
=
\begin{bmatrix}
\sigma + \frac{h}{2} f(x_0) \\ f(x_1) \\ \vdots \\ \vdots \\ f(x_m) \\ \beta
\end{bmatrix}
$$

Comment on order $\mathcal{O}(h^2)$

Same matrix as forward difference

But 1^{st} entry rhs different "double influence."

Case (iii): Second order right-sided difference.

$$u'(x_j) = \frac{1}{2h}\left[-3v_j + 4v_{j+1} - v_{j+2}\right] + \mathcal{O}(h^2).$$

So in our case, $j=0$, the approx. is

$$\frac{1}{h}\left[-\frac{3}{2}v_0 + 2v_1 - \frac{1}{2}v_2\right] = \sigma$$

Then, the new linear system is

$$\frac{1}{h^2}\begin{bmatrix} -\frac{3h}{2} & 2h & -h/2 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -2 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & \ddots & & & \vdots \\ \vdots & & & \ddots & \ddots & \ddots & & \vdots \\ & & & & & 1 & -2 & 1 \\ 0 & & & & & & 0 \cdots & h^2 \end{bmatrix}\begin{bmatrix} v_0 \\ v_1 \\ \vdots \\ \vdots \\ \vdots \\ v_m \\ v_{m+1} \end{bmatrix} = \begin{bmatrix} \sigma \\ f(x_1) \\ \vdots \\ \vdots \\ \vdots \\ f(x_m) \\ \beta \end{bmatrix}$$

Comment on convenience to use this or (ii) to obtain $\mathcal{O}(h^2)$.