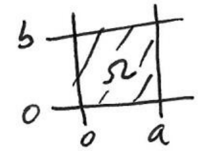
SOR Applied to Poisson's Equation.

BVP: $\begin{cases} \nabla^2 u = f(x,y), & (x,y) \in \Omega, \\ u(\vec{x}_s) = \gamma(\vec{x}_s), & \vec{x}_s \in \partial\Omega \end{cases}$



Using 5-point stencil finite difference method (2nd order centered fdm).

$$\boxed{\nabla_5^2 U_{jk} = f_{ij}} \qquad (0.1)$$

$$\boxed{\frac{U_{j+1,k} - 2U_{j,k} + U_{j-1,k}}{(\Delta x)^2} + \frac{U_{j,k+1} - 2U_{jk} + U_{j,k-1}}{\Delta y^2} = f_{jk}}$$

In matrix form: $\boxed{A\vec{U} = \vec{f}}$

Solving for U_{jk}, $j = 1, .., N_x - 1$, $k = 1, .., N_y - 1$ (1.1)

$$\boxed{U_{jk} = \theta_x \left(U_{j+1,k} + U_{j-1,k} \right) + \theta_y \left(U_{j,k+1} + U_{j,k-1} \right) - \theta_f f_{jk}}$$

(1.2)

where $\theta_x = \dfrac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}$, $\theta_y = \dfrac{\Delta x^2}{2(\Delta x^2 + \Delta y^2)}$

and

$$\theta_f = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

If $\Delta x = \Delta y \Rightarrow \theta_x = \frac{1}{4}, \quad \theta_y = \frac{1}{4}$

$$\theta_f = \frac{\Delta x^2}{4}$$

Thus,

$$\boxed{U_{jk} = \frac{U_{j+1,k} + U_{j-1,k} + U_{j,k+1} + U_{j,k-1}}{4} - \frac{\Delta x^2}{4} f_{jk}} \quad (2.1)$$

Equations (1.2) and (2.1) Can be written in matrix form as

$$\boxed{\vec{U} = M_j \vec{U} + \vec{C}} \qquad (2.2)$$

where $M_j = D^{-1}(L + UM)$

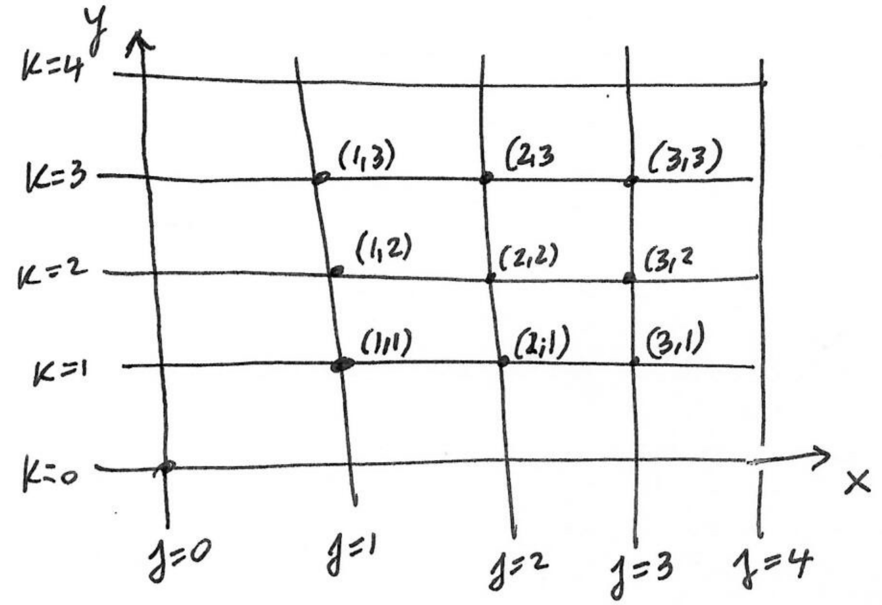
and $D = \begin{pmatrix} 1 & & \\ & 1 & \\ & & \ddots & \\ & & & 1 \end{pmatrix} = I$

$$L + UM = M_j$$

Since $D^{-1} = I$ exists then

$$A\vec{U} = \vec{f} \Leftrightarrow U = M_j \cdot \vec{U} + \vec{C}.$$

For $\underline{N_x = 4, \ N_y = 4}$ $\underline{\text{DIRICHLET PROBLEM}}$



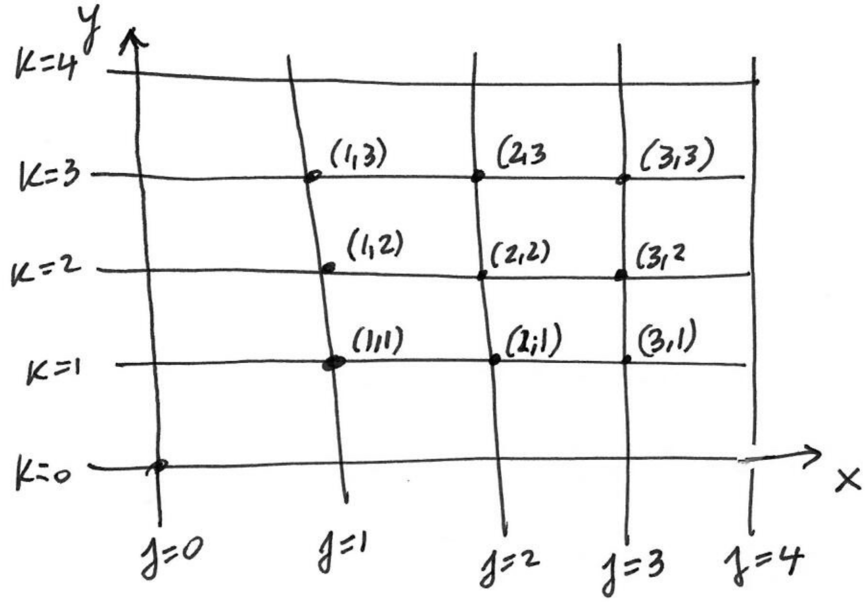
$$
A = \left(\begin{array}{c|ccc|ccc|ccc|c|c}
 & & & & & & & & & & & \\
K=1 & -1 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 & 0 & j=1 & v_{11} \\
 & \theta_x & -1 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 & j=2 & v_{21} \\
 & 0 & \theta_x & -1 & 0 & 0 & \theta_y & 0 & 0 & 0 & j=3 & v_{31} \\
K=2 & \theta_y & 0 & 0 & -1 & \theta_x & 0 & \theta_y & & 0 & j=1 & v_{12} \\
 & 0 & \theta_y & 0 & \theta_x & -1 & \theta_x & 0 & \theta_y & 0 & j=2 & v_{22} \\
 & 0 & 0 & \theta_y & 0 & \theta_x & -1 & 0 & 0 & \theta_y & j=3 & v_{32} \\
K=3 & 0 & 0 & 0 & \theta_y & 0 & 0 & -1 & \theta_x & 0 & j=1 & v_{13} \\
 & 0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & -1 & \theta_x & j=2 & v_{23} \\
 & 0 & 0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & -1 & j=3 & v_{33}
\end{array}\right)
$$

A $\qquad\qquad\qquad\qquad$ $\vec{v}$

Where

$$\vec{v} = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{N_x-1,1} \\ v_{12} \\ \vdots \\ v_{N_x-1,2} \\ \vdots \\ v_{1,N_y-1} \\ \vdots \\ v_{N_x-1,N_y-1} \end{bmatrix}, \qquad \vec{C} = -\theta_f \begin{bmatrix} f_{11} \\ \vdots \\ f_{N_x-1,1} \\ f_{12} \\ \vdots \\ f_{N_x-1,2} \\ \vdots \\ f_{1,N_y-1} \\ \vdots \\ f_{N_x-1,N_y-1} \end{bmatrix} \qquad (a.1)$$

For $N_x = 4,\ N_y = 4$ DIRICHLET PROBLEM



(10.1)

$$
\begin{array}{cc}
 & \\
\begin{array}{c} K=1 \\ \\ \\ M= \\ K=2 \\ \\ \\ K=3 \\ \\ \end{array} &
\left(\begin{array}{ccc}
\begin{pmatrix} 0 & \theta_x & 0 \\ \theta_x & 0 & \theta_x \\ 0 & \theta_x & 0 \end{pmatrix} & \begin{pmatrix} \theta_y & 0 & 0 \\ 0 & \theta_y & 0 \\ 0 & 0 & \theta_y \end{pmatrix} & \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
\begin{pmatrix} \theta_y & 0 & 0 \\ 0 & \theta_y & 0 \\ 0 & 0 & \theta_y \end{pmatrix} & \begin{pmatrix} 0 & \theta_x & 0 \\ \theta_x & 0 & \theta_x \\ 0 & \theta_x & 0 \end{pmatrix} & \begin{pmatrix} \theta_y & & 0 \\ 0 & \theta_y & 0 \\ 0 & 0 & \theta_y \end{pmatrix} \\
\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & \begin{pmatrix} \theta_y & 0 & 0 \\ 0 & \theta_y & 0 \\ 0 & 0 & \theta_y \end{pmatrix} & \begin{pmatrix} 0 & \theta_x & 0 \\ \theta_x & 0 & \theta_x \\ 0 & \theta_x & 0 \end{pmatrix}
\end{array}\right)
\begin{array}{c} j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3 \end{array}
\end{array}
\qquad (10.2)
$$

Iterative techniques:

$K = 1, \dots, N_x - 1$

$j = 1, \dots, N_y - 1$

a) Jacobi:

$$\boxed{U_{jk}^{(l+1)} = \theta_x \left(U_{j+1,k}^{(l)} + U_{j-1,k}^{(l)} \right) + \theta_y \left(U_{j,k+1}^{(l)} + U_{j,k-1}^{(l)} \right) + \theta_f f_{jk}} \qquad (5.1)$$

b) Gauss-Seidel:

$$\boxed{U_{jk}^{(l+1)} = \theta_x \left(U_{j+1,k}^{(l)} + U_{j-1,k}^{(l+1)} \right) + \theta_y \left(U_{j,k+1}^{(l)} + U_{j,k-1}^{(l+1)} \right) + \theta_f f_{jk}} \qquad (5.2)$$

c) SOR:

$$\boxed{U_{jk}^{(l+1)} = \omega \left(U_{jk}^{(l+1)} \right)_{GS} + (1-\omega) U_{jk}^{(l)}} \qquad (5.3)$$

In matrix form, as a fixed point problem:

$$\vec{U} = M_\omega \vec{U} + C_\omega$$

Key questions:

i) Conditions on original matrix A for converg.

ii) Appropriate range of ω for convergence.

iii) Finding optimum ω.

iv) Rate of convergence.

Now, we will find conditions on A and ω to guarantee convergence of SOR.

First, we need the following result:

Theorem.

For any matrix A $n \times n$

$$\det(A) = \prod_{i=1}^{n} \lambda_i$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A.

Proof. Recall

$$p(\lambda) = \det(\lambda I - A)$$

is the characteristic polynomial of A.

and

$$p(\lambda) = \lambda^n + \ldots + a_0 = (\lambda - \lambda_1) \ldots (\lambda - \lambda_n),$$

then,

$$\det(-A) = p(0) = (-1)^n \prod_{i=1}^{n} \lambda_i$$

But, $\det(-A) = (-1)^n \det(A)$

$$\Rightarrow (-1)^n \det(A) = \det(-A) = (-1)^n \prod_{i=1}^{n} \lambda_i \Rightarrow \det(A) = \prod_{i=1}^{n} \lambda_i \checkmark$$

Theorem (Kaham)

If $a_{ii} \neq 0$, for all i, then

If SOR converges $\Rightarrow$ $0 < \omega < 2$

Proof:

Since SOR converges $\Rightarrow \rho(M_\omega) < 1$ (8.1)

Also,

$$M_\omega = (D - \omega L)^{-1}\left(\omega U_M + (1-\omega)D\right) \quad (8.2)$$

where

$$A = \underbrace{D}_{\text{diag}} - \underbrace{L}_{\text{lower triang.}} - \underbrace{U_M}_{\text{upper triang.}}$$

Using that

$$\det(M_\omega) = \prod_{i=1}^{n} \lambda_i \quad \text{Eigenvalues of } M_\omega.$$

We can establish a relationship between (8.1) and (8.2).

In fact,

$$|M_\omega| = \left|(D-\omega L)^{-1}\left(\omega U_m + (1-\omega)D\right)\right|$$

$$= \left|(D-\omega L)^{-1}\right| \left|\omega U_m + (1-\omega)D\right|$$

Both are triangular matrices

$$= |D^{-1}|\,|D|\,(1-\omega)^n = (1-\omega)^n$$

On the other hand,

$$|M_\omega| = \prod_{i=1}^{n} \lambda_i$$

Therefore, $(1-\omega)^n = \prod_{i=1}^{n} \lambda_i$

$$\Rightarrow \quad |(1-\omega)|^n = \prod_{i=1}^{n} |\lambda_i| \leq \max |\lambda_k|^n = \rho^n(M_\omega)$$

$$\Rightarrow \quad |\omega-1|^n = |(1-\omega)|^n \leq \rho^n(M_\omega) < 1$$

$$\Rightarrow \quad (\omega-1) < 1 \Rightarrow \quad \underline{\underline{0 < \omega < 2}}$$

The condition $0 < \omega < 2$ is also sufficient for special set of matrices A.

Thm. (Ostrowski – Reich).

If A is positive definite and $\omega \in (0,2)$ then, SOR converges for any choice of initial approximation $\vec{v}^{(0)}$.

Recall A is positive definite if $\vec{x}^T A \vec{x} > 0$ for any $\vec{x} \neq \vec{0}$, and A is symmetric.

Also, A is P.D $\Longleftrightarrow$ All eigenvalues of A are positive.

Thm. If A is P.D. and tridiagonal, then $\rho(M_g) = \rho^2(M_j) < 1$ and the optimal choice of ω for SOR method is

$$\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho^2(M_j)}}$$

and $\rho(M_\omega) = \omega - 1$.

Notice $1 < \omega_{opt} < 2$

Application of previous theorem to our linear system.

$$A = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 3 & -1 \\ 0 & -1 & 2 \end{bmatrix}$$

A is P.D since $|2| = 2 > 0$, $\begin{vmatrix} 2 & -1 \\ -1 & 3 \end{vmatrix} = 5 > 0$ and $|A| = 12 - 4 = 8 > 0$. and it's also tridiag. then the prev. Thm. can be applied to A.

$$A = D - L - U = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 2 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} - \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

$$A\vec{x} = \vec{b} \Rightarrow D\vec{x} = (L+U)\vec{x} + \vec{b} \Rightarrow \vec{x} = D^{-1}(L+U)\vec{x} + D^{-1}\vec{b}$$

or $\boxed{\vec{x} = M_j \vec{x} + \vec{C}_j}$, $M_j \equiv D^{-1}(L+U)$.

For our example,

$$M_j = \underbrace{\begin{bmatrix} 1/2 & 0 & 0 \\ 0 & 1/3 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}}_{D^{-1}} \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}_{(L+U)} = \begin{bmatrix} 0 & 1/2 & 0 \\ 1/3 & 0 & 1/3 \\ 0 & 1/2 & 0 \end{bmatrix}$$

$$\Rightarrow |M_j - \lambda I| = \begin{vmatrix} -\lambda & 1/2 & 0 \\ 1/3 & -\lambda & 1/3 \\ 0 & 1/2 & -\lambda \end{vmatrix} = -\lambda^3 + \frac{2}{6}\lambda = 0 \Rightarrow \lambda\left(\lambda^2 - \frac{2}{6}\right) = 0$$

$$\Rightarrow \lambda_1 = 0, \quad \lambda_{2,3} = \pm \frac{\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\sqrt{3}} = \pm\frac{\sqrt{3}}{3}$$

$$\Rightarrow \rho(M_j) = \frac{\sqrt{3}}{3} \simeq 0.58.$$

$\Rightarrow$ Optimal ω, $\quad \omega_{opt} = \dfrac{2}{1 + \sqrt{1 - \rho^2(M_j)}} \simeq \underline{\underline{1.1.}}$ ✓

The iterative process

$$\vec{U}^{(k)} = M_j \vec{U}^{(k-1)} + \vec{C}$$

where M_j is given by (10.2) Corresponds to Jacobi's iterative method for this Poisson's problem.

It can be proved that the eigenvalues of M_j are given by

$$\boxed{\mu_{m,n} = 1 - 4\theta_x \sin^2\left(\frac{m\pi}{2N_x}\right) - 4\theta_y \sin^2\left(\frac{n\pi}{2N_y}\right)} \quad (11.1)$$

$m = 1, \ldots, N_x - 1$

$n = 1, \ldots, N_y - 1.$ → Go to R12 Converg. Theorems

$$\theta_x = \frac{1}{2}\frac{\Delta y^2}{\Delta x^2 + \Delta y^2}, \qquad \theta_y = \frac{1}{2}\frac{\Delta x^2}{\Delta x^2 + \Delta y^2}$$

If domain is square: $N_x = N_y$, $\Delta x = \Delta y$.

$$\Rightarrow \quad \theta_x = \frac{1}{4}, \quad \theta_y = \frac{1}{4}$$

$$\Rightarrow \boxed{\mu_{m,n} = 1 - \sin^2\left(\frac{m\pi}{2N_x}\right) - \sin^2\left(\frac{n\pi}{2N_x}\right)} \quad (11.2)$$

Largest eigenvalue when $m = n = 1$

$$\Rightarrow \boxed{\mu_{1,1} = 1 - 2\sin^2\left(\frac{\pi}{2N_x}\right)} \quad (11.3)$$

Obviously,

$$\boxed{|\mu_{1,1}| < 1 \Rightarrow \rho(M_j) < 1}$$

Therefore, Jacobi's iterative method converges to the fixed point $\vec{v}^*$ of

$$\vec{v} = M_J \vec{v} + \vec{c}, \quad \text{for any initial guess } \vec{v}_0.$$

This $\vec{v}^*$ is also the Solution of $A\vec{v} = \vec{f}$

Since the matrix D associated to this Jacobi's method for Laplace or Poisson equations is invertible,

since $$a_{ii} = -2\left(\frac{\Delta y^2 + \Delta x^2}{\Delta x^2 \Delta y^2}\right), \quad \text{for all } i.$$

Similar result can be proved even if domain is not a square and if $\Delta x \neq \Delta y$.

Using Taylor expansion for $\sin(x)$, when N_x large

$$\sin\left(\frac{\pi}{2N_x}\right) = \frac{\pi}{2N_x} - \frac{1}{3!}\left(\frac{\pi}{2N_x}\right)^3 + \cdots \quad (12.1)$$

and substituting this expression into (11.3)

$$\boxed{\rho(M_j) \simeq 1 - \frac{\pi^2}{2N_x^2}} \quad (12.2)$$

Therefore, $\rho(M_j) \simeq 1$, for large N_x.

This estimate serves to analize convergence.

Notice,

$$\|\vec{v}^{(k)} - \vec{v}^*\| \le \|M_j\|^k \|\vec{v}^{(0)} - \vec{v}^*\|$$

So if $\|M_j\| \simeq 1$ the convergence is very slow.

Also,

$$\rho(M_j) \le \|M_j\| \le \rho(M_j) + \varepsilon,$$

ε small.

So $\rho(M_j) \simeq 1$ also implies slow convergence.

<u>Relationship between $\rho(M_j)$ and $\rho(M_g)$ for Jacobi and Gauss-Seidel iterative techniques applied to Poisson's equation in a rectangular domain</u>

It can be proved

$$\rho(M_g) = \rho^2(M_j)$$

Then, using (12.2) in a square mesh

$$\rho(M_g) \simeq 1 - \frac{\pi^2}{N_x^2}$$

Therefore, G-S. iterative method is a little bit faster than Jacobi's method for Laplace or Poisson's equ. on a square mesh.

The <u>Ostrowsky - Reich theorem and the one after it in page 8, can be extended to some more general matrices A.</u> They include the pentadiagonal matrix A corresponding to the centered-finite difference for Poisson's equation.

It means

$$\vec{U}^{(k)} = M_\omega \vec{U}^{(k-1)} + \vec{C}_\omega$$

also converges to $\vec{U}^*$ soln. of both

$$\vec{U} = M_\omega \vec{U} + \vec{C}_\omega \quad \text{and} \quad A\vec{U} = \vec{f}.$$

for $\omega \in (0,2)$ and for any initial guess $\vec{U}^{(0)}$

also

$$\boxed{\omega_{opt} = \frac{2}{1+\sqrt{1-\rho^2(M_J)}}}$$

and

$$\boxed{\rho(M_{\omega_{opt}}) = \omega_{opt} - 1}$$

In our particular case of Poisson's equ. in a Square domain with $\Delta x = \Delta y$.

$$\rho(M_j) \simeq 1 - 2\sin^2\frac{\pi}{2N_x} = \cos\frac{\pi}{N_x}$$

Thus,
$$W_{opt} = \frac{2}{1+\sqrt{1-\cos^2\frac{\pi}{N_x}}} = \frac{2}{1+\sin\frac{\pi}{N_x}}$$

$\Rightarrow$
$$\rho(W_{opt}) = W_{opt} - 1 = \frac{1-\sin\frac{\pi}{N_x}}{1+\sin\frac{\pi}{N_x}}$$

Now,
$$\frac{2}{1+\sin\frac{\pi}{J}} \simeq 2\left(1-\sin\frac{\pi}{N_x}\right) \simeq 2\left(1-\frac{\pi}{N_x}\right), \text{ for large } N_x$$

$\Rightarrow$
$$\rho(W_{opt}) \simeq \left(2 - 2\frac{\pi}{N_x}\right) - 1 = 1 - \frac{2\pi}{N_x}, \text{ for large } N_x$$

Comparing the Spectral radius for the different iterative techniques applied to Poisson's equ. in a square with $\Delta x = \Delta y$.

$$\rho(M_j) \simeq 1 - \frac{\pi^2}{2N_x^2},$$

$$\rho(M_g) \simeq 1 - \frac{\pi^2}{N_x^2}, \text{ for large } N_x.$$

$$\rho(M_{w_{opt}}) \simeq 1 - \frac{\pi}{N_x},$$

From these asymptotic estimates for large N_x, it is seen that SOR converges much faster than Jacobi or G-S. iterative techniques for Poisson's equ. in a square domain with $\Delta x = \Delta y$.

Computation of ω_{opt} when $\Delta x = \Delta y$ but the domain Ω is not a square $N_x \neq N_y$.

In this case,

$$\rho(M_J) = 1 - \sin^2 \frac{\pi}{2N_x} - \sin^2 \frac{\pi}{2N_y}$$

$$= 1 - \frac{1}{2}\left(-\cos\frac{\pi}{N_x} + 1 - \cos\frac{\pi}{N_y} + 1\right)$$

$$= \frac{1}{2}\left(\cos\frac{\pi}{N_x} + \cos\frac{\pi}{N_y}\right)$$

Thus,

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho^2(M_J)}} = \frac{2}{1+\sqrt{1-\frac{1}{4}\left(\cos\frac{\pi}{N_x}+\cos\frac{\pi}{N_y}\right)^2}}$$

or

$$\boxed{\omega_{opt} = \frac{4}{2+\sqrt{4-\left(\cos\frac{\pi}{N_x}+\cos\frac{\pi}{N_y}\right)^2}}} \quad (16.1)$$

Summary

Finding the ω optimum ω_{OPT} for SOR applied to Poisson equation.

(I) Poisson Equ. (2D): $\boxed{u_{xx} + u_{yy} = f(x,y)}$

(II) 5-point scheme:

$$\boxed{\frac{v_{j,k+1} - 2v_{jk} + v_{j,k-1}}{(\Delta y)^2} + \frac{v_{j+1,k} - 2v_{jk} + v_{j-1,k}}{(\Delta x)^2} = f_{jk}}$$

(III) Direct Scheme:

$$\boxed{\theta_y v_{j,k-1} + \theta_x v_{k-1,j} - v_{jk} + \theta_x v_{k+1,j} + \theta_y v_{j,k+1} = \theta_f f_{jk}}$$

where

$$\theta_y = \frac{(\Delta x)^2}{2(\Delta x^2 + \Delta y^2)}, \quad \theta_x = \frac{\Delta y^2}{2(\Delta x^2 + \Delta y^2)}, \quad \theta_f = \frac{\Delta x^2 \Delta y^2}{2(\Delta x^2 + \Delta y^2)}$$

(IV) Fixed-Point Problem:

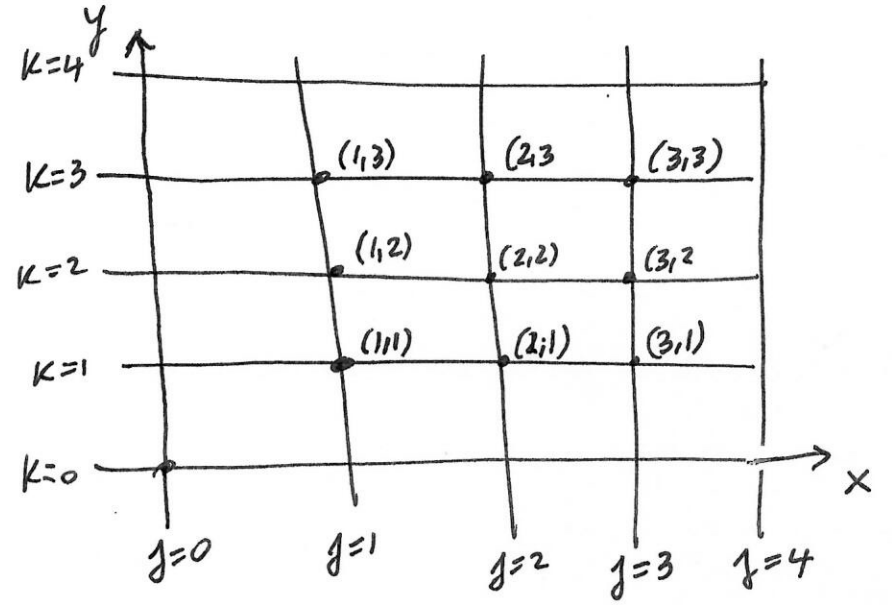
$$\boxed{v_{jk} = \theta_x \left(v_{j+1,k} + v_{j-1,k} \right) + \theta_y \left(v_{j,k+1} + v_{j,k-1} \right) - \theta_f f_{jk}}$$

or

$$\boxed{\vec{v} = M\vec{v} + \vec{C}}$$

where M is defined as follows assuming $N_x = 4 = N_y$.

For $N_x = 4$, $N_y = 4$ DIRICHLET PROBLEM



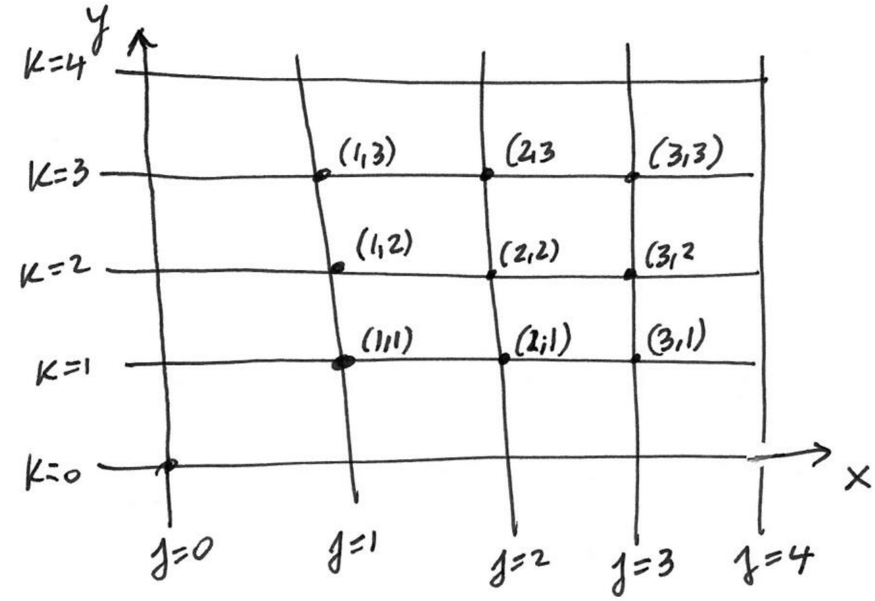
$$
\begin{array}{r}
\\
k=1 \\
\\
\\
A = \; k=2 \\
\\
\\
k=3 \\
\\
\end{array}
\left(\begin{array}{ccc|ccc|ccc}
-1 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 & 0 \\
\theta_x & -1 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 \\
0 & \theta_x & -1 & 0 & 0 & \theta_y & 0 & 0 & 0 \\
\hline
\theta_y & 0 & 0 & -1 & \theta_x & 0 & \theta_y & & 0 \\
0 & \theta_y & 0 & \theta_x & -1 & \theta_x & 0 & \theta_y & 0 \\
0 & 0 & \theta_y & 0 & \theta_x & -1 & 0 & 0 & \theta_y \\
\hline
0 & 0 & 0 & \theta_y & 0 & 0 & -1 & \theta_x & 0 \\
0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & -1 & \theta_x \\
0 & 0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & -1
\end{array}\right)
\begin{array}{l}
j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3
\end{array}
\begin{pmatrix}
v_{11} \\ v_{21} \\ v_{31} \\ v_{12} \\ v_{22} \\ v_{32} \\ v_{13} \\ v_{23} \\ v_{33}
\end{pmatrix}
$$

A $\qquad\qquad\qquad\qquad$ $\vec{v}$

Where

$$\vec{v} = \begin{bmatrix} v_{11} \\ v_{21} \\ \vdots \\ v_{N_x-1,1} \\ v_{12} \\ \vdots \\ v_{N_x-1,2} \\ \vdots \\ v_{1,N_y-1} \\ \vdots \\ v_{N_x-1,N_y-1} \end{bmatrix}, \qquad \vec{C} = -\theta_f \begin{bmatrix} f_{11} \\ \vdots \\ f_{N_x-1,1} \\ f_{12} \\ \vdots \\ f_{N_x-1,2} \\ \vdots \\ f_{1,N_y-1} \\ \vdots \\ f_{N_x-1,N_y-1} \end{bmatrix} \qquad (9.1)$$

For $N_x = 4$, $N_y = 4$ DIRICHLET PROBLEM



(10.1)

$$
\begin{array}{c}
K=1 \\ \\ \\ \\ M = \\ K=j \\ \\ \\ K=3
\end{array}
\left(
\begin{array}{ccc|ccc|ccc}
0 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 & 0 \\
\theta_x & 0 & \theta_x & 0 & \theta_y & 0 & 0 & 0 & 0 \\
0 & \theta_x & 0 & 0 & 0 & \theta_y & 0 & 0 & 0 \\
\hline
\theta_y & 0 & 0 & 0 & \theta_x & 0 & \theta_y & & 0 \\
0 & \theta_y & 0 & \theta_x & 0 & \theta_x & 0 & \theta_y & 0 \\
0 & 0 & \theta_y & 0 & \theta_x & 0 & 0 & 0 & \theta_y \\
\hline
0 & 0 & 0 & \theta_y & 0 & 0 & 0 & \theta_x & 0 \\
0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & 0 & \theta_x \\
0 & 0 & 0 & 0 & 0 & \theta_y & 0 & \theta_x & 0
\end{array}
\right)
\begin{array}{l}
j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3 \\ j=1 \\ j=2 \\ j=3
\end{array}
\qquad (10.2)
$$

Three Possible Iterative Techniques.

(1) $\vec{v}^{(l+1)} = M_J \vec{v}^{(l)} + \vec{C}$

(2) $\vec{v}^{(l+1)} = M_{gs} \vec{v}^{(l)} + \vec{C}_{gs}$

(3) $\vec{v}^{(l+1)} = M_\omega \vec{v}^{(l)} + \vec{C}_s$

Iterative techniques:

$K = 1, \dots, N_x - 1$

$j = 1, \dots, N_y - 1$

a) Jacobi:

$$\boxed{U_{jk}^{(\ell+1)} = \theta_x \left(U_{j+1,k}^{(\ell)} + U_{j-1,k}^{(\ell)} \right) + \theta_y \left(U_{j,k+1}^{(\ell)} + U_{j,k-1}^{(\ell)} \right) + \theta_f f_{jk}} \qquad (5.1)$$

b) Gauss-Seidel:

$$\boxed{U_{jk}^{(\ell+1)} = \theta_x \left(U_{j+1,k}^{(\ell)} + U_{j-1,k}^{(\ell+1)} \right) + \theta_y \left(U_{j,k+1}^{(\ell)} + U_{j,k-1}^{(\ell+1)} \right) + \theta_f f_{jk}} \qquad (5.2)$$

c) SOR:

$$\boxed{U_{jk}^{(\ell+1)} = \omega \left(U_{jk}^{(\ell+1)} \right)_{GS} + (1-\omega) U_{jk}^{(\ell)}} \qquad (5.3)$$

In matrix form, as a fixed point problem:

$$\vec{U} = M_\omega \vec{U} + C_\omega$$

Key questions:
i) Conditions on original matrix A for Converg.
ii) Appropriate range of ω for convergence.
iii) Finding optimum ω.
iv) Rate of convergence.

(Kaham)

(V) Range of ω: If $a_{ii} \neq 0$ (which is our case $a_{ii} = 1$)
Poisson's Equ.

then If SOR converges $\Rightarrow$ $0 < \omega < 2$.

(VI) Convergence SOR:

$\boxed{\vec{v}^{(k)} = M_\omega \vec{v}^{(k-1)} + \vec{C}_\omega}$ converges to $\vec{v}^*$

fixed point of $\vec{v} = M_\omega \vec{v} + \vec{c}$. for $0 < \omega < 2$

and for any initial Guess.

(VII) ω optimum of SOR and Spectral radius $\rho(M_{\omega_{opt}})$.

$$\boxed{\omega_{opt} = \frac{2}{1 + \sqrt{1 - \rho^2(M_j)}}}$$

and $\boxed{\rho(M_{\omega_{opt}}) = \omega_{opt} - 1}$

(VIII) Comparison of spectral radius

$\boxed{\rho(M_j) \simeq 1 - \frac{\pi^2}{2N_x^2}}$, $\boxed{\rho(M_g) \simeq 1 - \frac{\pi^2}{N_x^2}}$

$$\boxed{\rho(M_{\omega_{opt}}) \simeq 1 - \frac{\pi}{N_x}}$$

Computation of $\rho(M_j)$

(IV) Eigenvalues of M_j

$$\boxed{\mu_{mn} = 1 - 4\theta_x \sin^2\left(\frac{m\pi}{2N_x}\right) - 4\theta_y \sin^2\left(\frac{n\pi}{2N_y}\right)}$$

$m = 1, \dots, N_x - 1$
$n = 1, \dots, N_y - 1$

If $N_x = N_y$, $\Delta x = \Delta y$.

$$\boxed{\mu_{mn} = 1 - \sin^2\left(\frac{m\pi}{2N_x}\right) - \sin^2\left(\frac{n\pi}{2N_x}\right)}$$

(V) Largest Eigenvalue: When $m = n = 1$

$$\boxed{\mu_{1,1} = 1 - 2\sin^2\left(\frac{\pi}{2N_x}\right)}$$

W_{opt} when $\Delta x = \Delta y$ but $N_x \neq N_y$

$$\boxed{\rho(M_j) = 1 - \sin^2\frac{\pi}{2N_x} - \sin^2\frac{\pi}{2N_y} = \frac{1}{2}\left(\cos\frac{\pi}{N_x} + \cos\frac{\pi}{N_y}\right)}$$

and

$$\omega_{opt} = \frac{2}{1+\sqrt{1-\rho^2(M_J)}} = \frac{2}{1+\sqrt{1-\frac{1}{4}\left(\cos\frac{\pi}{N_x}+\cos\frac{\pi}{N_y}\right)^2}}$$

or

$$\boxed{\omega_{opt} = \frac{4}{2+\sqrt{4-\left(\cos\frac{n\pi}{N_x}+\cos\left(\frac{\pi}{N_y}\right)\right)^2}}\ ds}$$