2.21 Spectral Methods

We have already seen finite difference approx. of differential operators as interpolation by local polynomials using few and closely enough interpolation points. (Grid points).

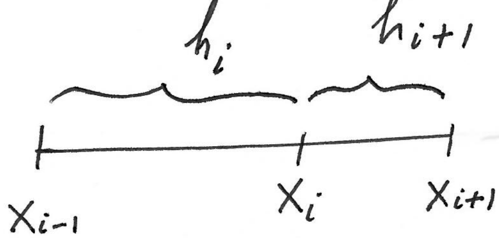
For example, given three points

x_{i-1}, x_i, x_{i+1} and their values U_{i-1} U_i. U_{i+1}

The divided difference Newton interpolation polynomial is given by

$$\boxed{\begin{aligned} p_i(x) = {} & U[x_{i-1}] + U[x_{i-1}, x_i]\,(x - x_i) \\ & + U[x_{i-1}, x_i, x_{i+1}]\,(x - x_{i-1})(x - x_i) \end{aligned}} \tag{1.1}$$

Assume



If h is uniform $h_i = h_{i+1}$

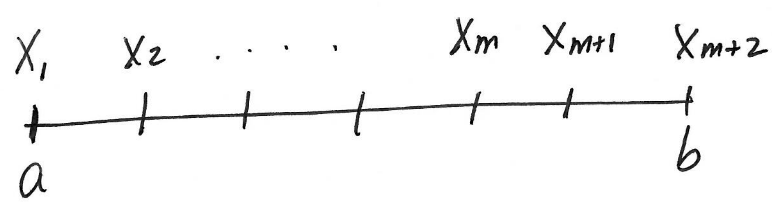
then,

$$p_i''(x_i) = U[x_1, x_2, x_3]$$

$$= \frac{1}{h^2} U_{i-1} - \frac{2}{h^2} U_i + \frac{1}{h^2} U_{i+1}$$

$$= D^2 U_i$$

If we consider a partition of $m+2$ points



we have m interior points (not counting boundaries)

So we have m local polynomials as (1.1)

Now, Suppose we want to obtain an approx. Soln. for

$$\begin{cases} u''(x) = f(x) \\ u(a) = \alpha, \quad u(b) = \beta \end{cases} \qquad (3.1)$$

and we use a partition of $m+2$ points and approximate the BVP (3.1) as

$$\begin{cases} p_i''(x_i) = D^2 U_i = f(x_i), & i = 2, .., m+1 \\ p(x_1) = \underset{U_1}{\overset{\shortparallel}{\alpha}}, \quad p_{m+}(x_{m+2}) = \beta = U_{m+2} & \end{cases}$$

This problem can be represented as the linear system

$$\boxed{A U = \vec{F}} \qquad (3.1)$$

By solving it all the $p_i(x)$ $(i=2, .., m+1)$ polynomials are obtained at once. from the discrete Soln: $U_1, .., U_m$.

Solution of (3.1) $\vec{v}$ satisfies (for uniform grid)

$$v_i = u(x_i) + \mathcal{O}(h^2)$$

Using interpolation polynomials $p_i(x)$ for the 3 points (x_{i-1}, v_{i-1}), (x_i, v_i), and (x_{i+1}, v_{i+1})

$i = 2, \ldots, m+1$.

With $p_1(x)$ and $p_{m+2}(x)$ special ones depending on BCs.

The approximation can be improved using polynomials interpolating more points.

For instance, if we use n points

$$v_i = u(x_i) + \mathcal{O}\left(h^{n-k}\right) \text{ in general.}$$

Natural extension is to consider stencils including all $m+2$ points in the grid.

Then, we have $(m+2)$ points and $(m+2)$ polynomials of degree at most $(m+1)$ interpolating the same $(m+2)$ points.

From the Fundamental theorem of Algebra

$$p_i(x) = p(x) \quad \text{for all } i = 1, \ldots, m+2.$$

It means there is one single polynomial of degree at most $(m+1)$ such that

$$p''(x_i) = u''(x_i) = f(x_i)$$

From the Fund. thm. of finite difference

$$U_i = u(x_i) + \mathcal{O}\left(h^{(m+2)-2}\right) \text{ for uniform grid}$$

$$\mathcal{O}(h^m).$$

Also,

$$h = \frac{1}{m+1} \underset{m \text{ large}}{\simeq} \frac{1}{m} \Rightarrow m \simeq 1/h$$

Then, for m large or equivalent for $h \to 0$.

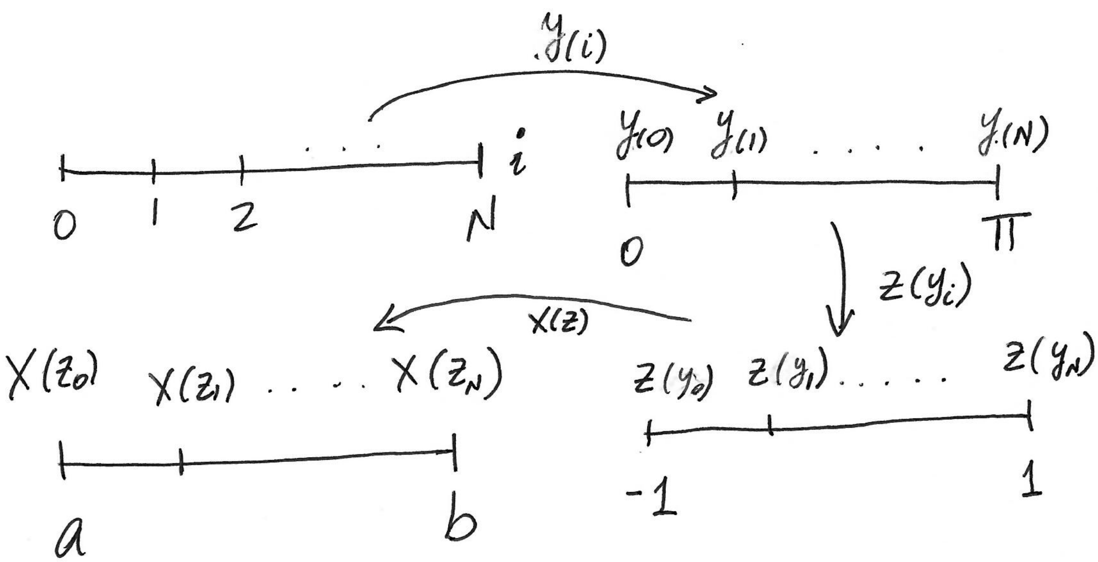
$$\mathcal{O}(h^m) = \mathcal{O}(h^{1/h})$$

And $h^{1/h} \to 0$ faster than any h^p for p fixed.

To avoid the oscillations that occur when interpolating points in a uniform grid by polynomials, it is recommended that we use a Chebyshev grid-type in this case.

Show simulation!

Chebyshev grid



Now, $y(i) = \frac{i\pi}{N}$, $z(y) = -\cos(y)$

$$X(z) = \left(\frac{b-a}{2}\right) z + \frac{b+a}{2} .$$

Then,

$$\boxed{X_i = X(z_i) = \frac{b+a}{2} - \left(\frac{b-a}{2}\right)\cos\left(\frac{i\pi}{N}\right)}$$