Stability. Maximum Principle.

Notice that the FT–CS difference scheme for heat conduction (homogeneous BCs) can be expressed in matrix form:

$$\begin{bmatrix} U_1^{n+1} \\ U_2^{n+1} \\ \vdots \\ U_{J-1}^{n+1} \end{bmatrix} = \begin{bmatrix} 1-2r & r & 0 & \cdots & 0 \\ r & 1-2r & r & 0 \cdots & 0 \\ & \ddots & \ddots & \ddots & \\ & & r & 1-2r & r \\ 0 & & & r & 1-2r \end{bmatrix} \begin{bmatrix} U_1^n \\ \vdots \\ \vdots \\ U_{J-1}^n \end{bmatrix}$$

or

$$\vec{U}^{n+1} = L_\Delta \vec{U}^n.$$

STABILITY

Definition: Consider two different initial value problems for the same finite difference scheme, i.e.,

$$\vec{U}^{n+1} = L_\Delta \vec{U}^n, \qquad \vec{U}^0 = \phi$$

$$\vec{V}^{n+1} = L_\Delta \vec{V}^n, \qquad \vec{V}^0 = \psi$$

This finite difference scheme is stable if there exists a positive constant C, independent of the mesh spacing an initial data, such that

$$\|\vec{U}^n - \vec{V}^n\| \le C \|\vec{U}^0 - \vec{V}^0\|, \quad n \to \infty, \ \Delta x \to 0, \ \Delta t \to 0, \ n\Delta t \le T.$$

If L_Δ is linear

$$\vec{U}^{n+1} - V^{n+1} = L_\Delta \vec{U}^n - L_\Delta \vec{V}^n = L_\Delta (\vec{U}^n - V^n)$$

Calling $\vec{v}^n = \vec{U}^n - \vec{V}^n$

$$\boxed{\vec{v}^{n+1} = L_\Delta v^n, \qquad \vec{v}^0 = \phi - \psi}$$

And in the definition of stability

$\|\vec{U}^n - V^n\| \leq c\,\|U^0 - V^0\|$ Can be Substituted

by $\boxed{\|\vec{v}^n\| \leq C\,\|v^0\|}$

Therefore,

If L_Δ is linear, the definition of stability can be written as

<u>Definition:</u> A finite difference scheme

$$\vec{U}^{n+1} = L_\Delta \vec{U}^n,$$ for a homogeneous IVP (initial value problem)

is stable if there exists a positive constant C, independent of the mesh spacing and initial data such that

$$\|\vec{U}^n\| \le C \|\vec{U}^0\|, \quad n \to \infty, \ \Delta x \to 0, \ \Delta t \to 0, \ n\Delta t \le T.$$

<u>Remark:</u> When L_Δ is linear the two definitions are equivalent.

<u>Maximum principle</u>

Finite difference schemes as the FT-CS and Crank-Nicholson for the heat equation are called one-level finite difference schemes because they only involve solutions at time levels "n" and "$n+1$".

Theorem.

A sufficient condition for stability of the one-level finite difference scheme

$$U_j^{n+1} = \sum_{|s| \le S} C_s U_{j+s}^n \qquad (\bullet)$$

in the $\|\cdot\|_\infty$ is that all coefficients C_s, $(|s| \le S)$, be positive and add to unity.

Proposition

The FT-CS finite difference scheme applied to a homogeneous IVP is stable if $r \le 1/2$

Proof.

$$U_j^{n+1} = r U_{j-1}^n + (1-2r) U_j^n + r U_{j+1}^n$$

then

$$\sum_{|s| \le S} C_s = r + (\overset{\ge 0 \text{ since } r \le 1/2}{1-2r}) + r = 1 \quad \checkmark$$

$$r = \sigma \frac{\Delta t}{\Delta x^2} > 0.$$

Proof of Theorem

Using triangular inequality in (•)

$$|U_j^{n+1}| \le \sum_{|s|\le S}' |c_s|\,|U_{j+s}^n| \,, \quad j = 1, 2, .., J-1$$

introducing $\|\cdot\|_\infty$

$$\|\vec{U}^{n+1}\| \le \underbrace{\Big(\sum_{|s|\le S}' c_s\Big)}^{=1} \|\vec{U}^n\| = \|\vec{U}^n\|$$

$$\Rightarrow \|\vec{U}^n\| \le \|\vec{U}^{n-1}\| \le \dots \le \|\vec{U}^0\|, \quad n \to \infty, \ \Delta x \to 0$$

$$\Delta t \to 0$$

$$n\Delta t \le T.$$

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what happens if

$$\sum_{|s|\le S}' c_s = C > 1 \quad ?$$

Order of numerical scheme

Definition: A consistent finite-difference scheme approximating a partial differential equation is of order p in time and order q in space if

$$T_j^n = \mathcal{O}(\Delta t^p) + \mathcal{O}(\Delta x^q).$$