2.15 A General Linear 2nd order Equation

$$\begin{cases} a(x)\,u''(x) + b(x)\,u'(x) + c(x)\,u(x) = f(x), & a < x < b \\ u(a) = \alpha, \quad u(b) = \beta \end{cases}$$

2nd order discret.

$$a_i \frac{v_{i+1} - 2v_i + v_{i-1}}{h^2} + b_i \frac{v_{i+1} - v_{i-1}}{2h} + c_i v_i = f(x_i)$$

$i = 1, \ldots, m$

It gives

$i=1$ $\left(-\frac{2a_1}{h^2} + c_1\right)v_1 + \left(\frac{a_1}{h^2} + \frac{b_1}{2h}\right)v_2 = f_1 - \left[\frac{a_1}{h^2} - \frac{b_1}{2h}\right]\overset{\alpha}{\overset{\shortparallel}{v_0}}$

$i=2$ $\left(\frac{a_2}{h^2} - \frac{b_2}{2h}\right)v_1 + \left(-\frac{2a_2}{h^2} + c_2\right)v_2 + \left(\frac{a_2}{h^2} + \frac{b_2}{2h}\right)v_3 = f_2$

$i=j$ $\left(\frac{a_j}{h^2} - \frac{b_j}{2h}\right)v_{j-1} + \left(-\frac{2a_j}{h^2} + c_j\right)v_j + \left(\frac{a_j}{h^2} + \frac{b_j}{2h}\right)v_{j+1} = f_j$

$i=m$

$\left(\frac{a_m}{h^2} - \frac{b_m}{2h}\right)v_{m-1} + \left(-\frac{2a_m}{h^2} + c_m\right)v_m = f_m - \left[\frac{a_m}{h^2} + \frac{b_m}{2h}\right]\overset{\beta}{\overset{\shortparallel}{v_{m+1}}}$

This leads to the linear system: $A\vec{U} = \vec{F}$

where

$$A = \frac{1}{h^2}\begin{bmatrix} c_1 h^2 - 2a_1 & a_1 + \frac{b_1 h}{2} & 0 & 0 & \cdots & 0 \\ a_2 - \frac{h b_2}{2} & h^2 c_2 - 2a_2 & a_2 - \frac{h b_2}{2} & 0 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \ddots & \ddots & \vdots \\ 0 & & & & a_m - \frac{h b_m}{2} & h^2 c_m - 2a_m \end{bmatrix}$$

$$U = [U_1 \; U_2 \ldots U_m]^T$$

$$\vec{F} = \left[f_1 - \left(\frac{a_1}{h^2} - \frac{b_1}{2h}\right)\alpha \;\; f_2 \ldots f_{m-1} \;\; f_m - \left(\frac{a_m}{h^2} + \frac{b_m}{2h}\right)\beta \right]^T$$

Example 2.1 Heat conduction in a rod with varying conductivity

Steady-state: $\boxed{[k(x)\,u'(x)]' = f(x)}$

If diff. $\boxed{k(x)\,u''(x) + k'(x)\,u'(x) = f(x)}$

And matrix equation for Centered discretization has

$$a_i = k_i, \quad b_i = k_i', \quad c_i \equiv 0$$

$$A = \frac{1}{h^2}\begin{bmatrix} -2k_1 & k_1 + \frac{hk_1'}{2} & 0 & 0 & \cdots & 0 \\ k_2 - \frac{hk_2'}{2} & -2k_2 & k_2 + \frac{hk_2'}{2} & 0 & \cdots & 0 \\ \vdots & & \ddots & & & \vdots \\ \vdots & & & \ddots & & \vdots \\ & & & & & -2k_m \\ 0 & & & k_m - \frac{hk_m'}{2} & & \end{bmatrix}$$

Nonsymmetric matrix

Instead, use centered difference approx. for

$$(ku')'(x_i)$$

with ½ step.

It means

$$(ku')'(x_i) \simeq \frac{(ku')_{i+1/2} - (ku')_{i-1/2}}{2\,\frac{h}{2}}$$

$$= \frac{k_{i+1/2}\,u'_{i+1/2} - k_{i-1/2}\,u'_{i-1/2}}{h} \quad (*)$$

And use centered diffce again with ½ step for $u'_{i+1/2}$ and $u'_{i-1/2}$

$$u'_{i+1/2} \simeq \frac{U_{i+1} - U_i}{2\frac{h}{2}} = \frac{U_{i+1} - U_i}{h}$$

$$u'_{i-1/2} \simeq \frac{U_i - U_{i-1}}{2\,h/2} = \frac{U_i - U_{i-1}}{h}$$

Subst. into (*)

$$(\kappa u')'(x_i) \simeq \frac{1}{h^2}\Big[\, \kappa_{i-1/2} U_{i-1} - (\kappa_{i+1/2} + \kappa_{i-1/2}) U_i + \kappa_{i+1/2} U_{i+1} \Big]$$

therefore the matrix A associated to this discretiz.

$$A = \frac{1}{h^2}\begin{bmatrix} -(\kappa_{1/2} + \kappa_{3/2}) & \kappa_{3/2} & 0 & 0 \cdots & 0 \\ \kappa_{3/2} & -(\kappa_{3/2} + \kappa_{5/2}) & \kappa_{5/2} & & \vdots \\ & \ddots & \ddots & \ddots & \\ & & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & \kappa_{m-1/2} & -(\kappa_{m-1/2} + \kappa_{m+1/2}) \end{bmatrix}$$

Symmetric matrix

- As it should be, because ODE is self-adjoint.
- Since $\kappa > 0 \Rightarrow$ matrix A nonsing. negative-definite $\Rightarrow$ all eigenv. negative

- Soln. to diffce equ. satisfies max pple for $f(x) \equiv 0$ $\quad U_i$ btw α and β.