Name: ____________________

Student ID: ____________________

Section: ____________________

Instructor: ____________________

Math 113 (Calculus 2)
Exam 1
25-29 January 2008

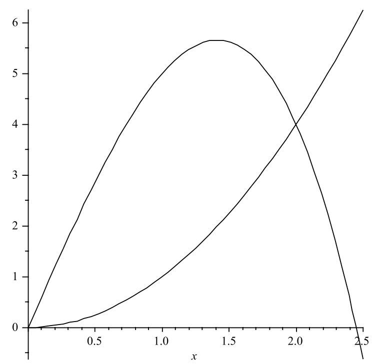
Instructions:

1. Work on scratch paper will not be graded.
2. Should you have need for more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
3. Simplify your answers. Expressions such as $\ln(1)$, e^0, $\sin(\pi/2)$, etc. must be simplified for full credit.
4. Calculators are not allowed.

For Instructor use only.

#	Possible	Earned		#	Possible	Earned
1.a	6			4	10	
1.b	6			5.a	8	
1.c	6			5.b	8	
1.d	6			5.c	8	
1.e	6			5.d	8	
2	10			5.3	8	
3	10			Total	100	

1. (30%) Consider the region between the curves $y = x^2$ and $y = 6x - x^3$ in the first quadrant.



(a) Set up an integral for the area of the region bounded by the curves. DO NOT EVALUATE.

$$\int_0^2 (6x - x^3 - x^2)\, dx$$

(b) Set up an integral for the volume when the region is rotated about the $x-$axis. DO NOT EVALUATE.

$$\pi \int_0^2 ((6x - x^3)^2 - (x^2)^2)\, dx$$

(c) Set up an integral for the volume when the region is rotated about the $y - axis$. DO NOT EVALUATE.

$$2\pi \int_0^2 x(6x - x^3 - x^2)\, dx$$

(d) Set up an integral for the volume when the region is rotated about the line $x = -1$. DO NOT EVALUATE.

$$2\pi \int_0^2 (x + 1)(6x - x^3 - x^2)\, dx$$

(e) Set up an integral for the volume when the region is rotated about the line $y = 6$. DO NOT EVALUATE.

$$\pi \int_0^2 ((6 - x^2)^2 - (6 - 6x + x^3)^2)\, dx$$

2. (10%) Use the disk method or the shell method to show that the volume V of a sphere with radius r is given by $V = \frac{4}{3}\pi r^3$.

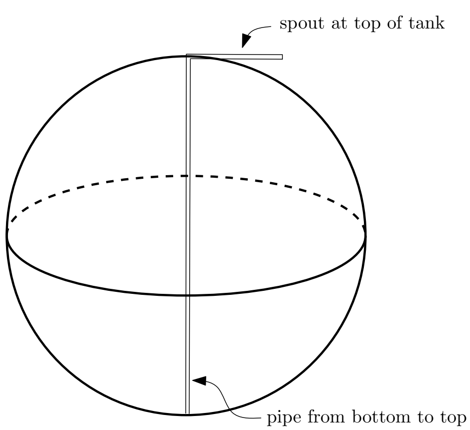
   The equation of a circle of radius r centered at the origin is given by $x^2 + y^2 = r^2$. Solving for y we get $y = \pm\sqrt{r^2 - x^2}$. We take the curve given by the positive square root and rotate the region under it about the x-axis from $0 \le x \le r$ to get the volume of a hemisphere. Multiplying by 2 gives the volume of the sphere.

$$V = 2\pi \int_0^r \left(\sqrt{r^2 - x^2}\right)^2 dx = 2\pi \int_0^r (r^2 - x^2) dx = 2\pi \left(r^2 x - \frac{x^3}{3}\right)\Big|_0^r = 2\pi(r^3 - \frac{r^3}{3}) = \frac{4\pi r^3}{3}$$

3. (10%) A heavy rope, 100 ft long, weighs 0.5 lb/ft and hangs over the edge of a building 120 ft high. How much work is done in pulling the rope to the top of the building?

$$\int_0^{100} x(.5)dx \quad \text{foot} - \text{pounds} = \frac{x^2}{4}\Big|_0^{100} \quad \text{foot} - \text{pounds} = 2,500 \quad \text{foot} - \text{pounds}$$

4. (10%) A spherical tank having radius 10 feet is filled with a fluid which weighs 100 pounds per cubic foot. This tank is half full. Find the work in foot pounds needed to pump the fluid out of a hole in the top of the tank.



Math 113 Exam 1 Problem 4

Solution 1. Regard the spherical tank as being obtained by rotating the circle $x^2+y^2=10^2$ about the y-axis. Then the center of the tank is at the origin and the liquid occupies the space in the tank with $-10\le y\le 0$ and the top of the tank is at $y=10$. The volume of a thin layer of liquid at height y is

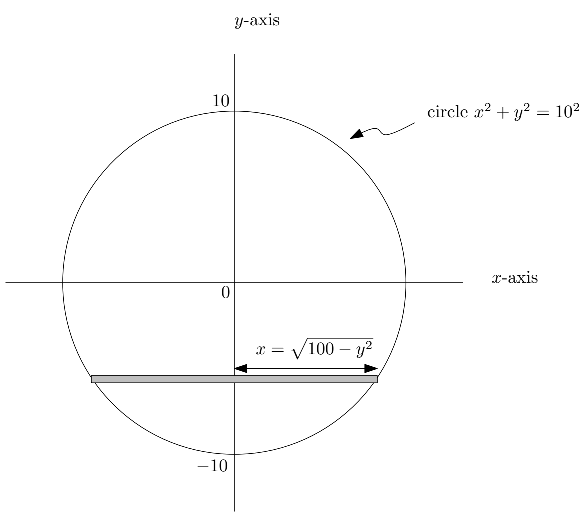
$$\text{volume of slice} = \pi x^2 dy = \pi\big(\sqrt{100-y^2}\big)^2 dy = \pi(100-y^2)\,dy.$$

Since the liquid weighs 100 pounds per cubic foot, the weight of the slice is

$$\text{weight of slice} = 100(\text{volume of slice}) = 100\pi(100-y^2)\,dy.$$

The distance from the thin layer at height y to the top of the tank is $10-y$, and there are thin layers for values of y such that $-10\le y\le 0$. Thus, the work W to empty the tank is

$$\begin{aligned}
W &= \int_{-10}^{0} 100\pi(100-y^2)(10-y)\,dy && \text{(6 points)}\\
&= 100\,\pi\int_{-10}^{0}\big(1000-100y-10y^2+y^2\big)\,dy = 100\,\pi\Big[1000y-\frac{100y^2}{2}-\frac{10y^3}{3}+\frac{y^4}{4}\Big]\Big|_{-10}^{0}\\
&= 0-100\,\pi\Big[-10000-\frac{10000}{2}+\frac{10000}{3}+\frac{10000}{4}\Big] && \text{(2 more points)}\\
&= 1{,}000{,}000\,\pi\Big[1+\frac{1}{2}-\frac{1}{3}-\frac{1}{4}\Big] = 1{,}000{,}000\,\pi\cdot\tfrac{11}{12} = 2{,}500{,}000\cdot\tfrac{11}{3}\cdot\pi = \boxed{\frac{2{,}750{,}000\,\pi}{3}} && \text{(2 more points)}
\end{aligned}$$



□

Grading Notes: If the solution had the wrong integral and used the formula $x^2+y^2=100$, the maximum score was 3/10. If solution had the wrong integral and didn't use the formula $x^2+y^2=100$ or take into account varying volumes and varying distances, the score was 0/10. If the integral was not set up correctly, the 4 points for the calculation were not awarded.

Solution 2. Place the origin of the coordinate system at the bottom of the circle. Then the center of the circle is at $y=10$, the top of the circle is at $y=20$, the equation for the circle is $x^2+(y-10)^2=100$, and the liquid is in the space $0\le y\le 10$. Again obtain the sphere by rotating the circle about the y-axis. In this case, an argument similar to that in the first solution gives

$$W = 100\pi\int_0^{10}(20-y)\big(\sqrt{10^2-(y-10)^2}\big)^2\,dy = 100\pi\int_0^{10}(20-y)\,\underbrace{\big[100-(y-10)^2\big]}_{x^2}\,dy = \cdots = \boxed{\frac{2{,}750{,}000\,\pi}{3}}$$

□

Common error in Solution 2: $W = 100\pi\displaystyle\int_0^{10}(20-y)\,\underbrace{(100-y^2)}_{\text{wrong}}\,dy = \frac{3{,}250{,}000\pi}{3}$

Solutions 3, 4, 5, 6. There are at least four more correct ways to set the problem up that people used. □

5. (40%) Evaluate the following integrals:

(a) $\int_0^\pi x \sin x \, dx$

Integration by parts. $u = x$, $dv = \sin x dx$, $du = dx$, and $v = -\cos x$

$$\int_0^\pi x \sin x \, dx = -x \cos x \Big|_0^\pi + \int_0^\pi \cos x dx = \pi + \sin x \Big|_0^\pi = \pi$$

(b) $\int_0^\pi \sin^2(2x) \, dx$

$$\int_0^\pi \sin^2(2x) \, dx = \int_0^\pi \frac{1 - \cos(4x)}{2} \, dx = \left(\frac{x}{2} - \frac{\sin(4x)}{8} \right) \Big|_0^\pi = \frac{\pi}{2}$$

(c) $\int (\ln x)^2 \, dx$

Integration by parts. $u = (\ln x)^2$, $dv = dx$, $du = 2\frac{\ln x}{x} dx$, $v = x$.

$$\int (\ln x)^2 \, dx = x(\ln x)^2 - 2 \int \ln x dx$$

Now $\int (\ln x)^2 \, dx = x \ln x - x + C$ by a second application of integration by parts. The final answer is $x(\ln x)^2 - 2x \ln x + 2x + C$

(d) $\int e^{3\theta} \cos(2\theta) \, d\theta$

Use Integration by parts $u = e^{3\theta}$, $dv = \cos(2\theta)d\theta$, $du = 3e^{3\theta}d\theta$, $v = \sin(2\theta)/2$. Let $I = \int e^{3\theta} \cos(2\theta) \, d\theta$.

$$I = \frac{e^{3\theta} \sin(2\theta)}{2} - \frac{3}{2} \int e^{3\theta} \sin(2\theta) d\theta$$

Use Integration by parts $U = e^{3\theta}$, $dV = \sin(2\theta)d\theta$, $dU = 3e^{3\theta}d\theta$, $V = -\cos(2\theta)/2$.

$$I = \frac{e^{3\theta} \sin(2\theta)}{2} - \frac{3}{2} \left(\frac{-e^{3\theta} \cos(2\theta)}{2} + \frac{3}{2} \int e^{3\theta} \cos(2\theta) d\theta \right) = \frac{(2 \sin(2\theta) + 3 \cos(2\theta))e^{3\theta}}{4} - \frac{9}{4} I$$

$$\frac{13}{4} I = \frac{(2 \sin(2\theta) + 3 \cos(2\theta))e^{3\theta}}{4} \text{ and } I = \frac{(2 \sin(2\theta) + 3 \cos(2\theta))e^{3\theta}}{13} + C$$

(e) $\int \sin^4 x \cos^3 x \ dx$

Let $u = \sin x$ so $du = \cos x dx$

$$\int \sin^4 x \cos^3 x \ dx = \int \sin^4 x \cos^2 x (\cos x dx) = \int \sin^4 x (1 - \sin^2 x)(\cos x dx)$$

$$= \int (u^4 - u^6) \ du = u^5/5 - u^7/7 + C = \sin^5 x/5 - \sin^7/7 + C$$