Name: ____________________

Student ID: ____________________

Section: ____________________

Instructor: ____________________

Math 113 (Calculus 2)
Exam 4

4–8 April 2008

Instructions:

1. Work on scratch paper will not be graded.
2. For question 1 and questions 10 through 15, show all your work in the space provided. Full credit will be given only if the necessary work is shown justifying your answer. Please write neatly.
3. Questions 2 through 9 are short answer. Fill in the blank with the appropriate answer. You do not need to show your work.
4. Should you have need for more space than is allotted to answer a question, use the back of the page the problem is on and indicate this fact.
5. Simplify your answers. Expressions such as $\ln(1)$, e^0, $\sin(\pi/2)$, etc. must be simplified for full credit.
6. Calculators are not allowed.

For Instructor use only.

#	Possible	Earned		#	Possible	Earned
1a	5			6	5	
1b	5			7	5	
1c	5			8	5	
1d	5			9	5	
1e	5			10	5	
1f	5			11	5	
2	5			12	5	
3	5			13	5	
4	5			14	5	
5	5			15	5	
				Total	100	

Unless indicated, each problem is worth 5%.

1. (30% Show your work.) Determine whether each series converges absolutely, converges conditionally, or fails to converge. State and justify your conclusion next to the series.

   (a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n \ln n}$

   Diverges by Integral Test. $\displaystyle\int_2^{\infty} \frac{1}{x \ln x} dx$ diverges.

   (b) $\displaystyle\sum_{n=1}^{\infty} \frac{\sin n}{n^2+1}$

   $\displaystyle\left|\frac{\sin n}{n^2+1}\right| \leq \frac{1}{n^2+1} \leq \frac{1}{n^2}$ Converges Absolutely by Comparison Test since $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p > 1$).

   (c) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^k k}{k^2+1}$

   $\displaystyle\frac{k}{k^2+1}$ is decreasing ($y = \displaystyle\frac{x}{x^2+1}$ is decreasing since $y' < 0$ for $x > 1$). Since the terms go to zero, the series converges by the Alternating Series Test. It does not converge absolutely by the Limit Comparison Test with the harmonic series. The series converges conditionally.

   (d) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2+1}}{n^2}$

   Use the Limit Comparison Test with $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n^2}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$. The series converges absolutely.

   (e) $\displaystyle\sum_{n=1}^{\infty} \frac{n!}{2^n}$

   Use the Ratio Test. $\displaystyle\lim_{n\to\infty} \frac{n+1}{2} = \infty$. So the series diverges.

   (f) $\displaystyle\sum_{k=1}^{\infty} \frac{(-1)^n \ln n}{n}$

   $\displaystyle\frac{\ln n}{n}$ decreases to zero for $n > 4$ ($y = \displaystyle\frac{\ln x}{x}$ is decreasing for $x > e$ since $y' < 0$ for $x > e$). Converges by Alternating Series Test. Doesn't converge absolutely by Comparison Test with the harmonic series. The series converges conditionally.

Questions 2 through 9 are short answer. Fill in the blank with the appropriate answer. You do not need to show your work.

2. If $f(x) = x^3 \sin x$, find the 100th derivative evaluated at zero; i.e., find $f^{(100)}(0)$.

   $100 \cdot 99 \cdot 98 = 970,200$

3. Evaluate the sum. $1 + \frac{1}{2 \cdot 1!} + \frac{1}{2^2 \cdot 2!} + \frac{1}{2^3 \cdot 3!} + \frac{1}{2^4 \cdot 4!} + \cdots = \sum_{k=0}^{\infty} \frac{1}{2^k \cdot k!}$

   $\sqrt{e}$

4. Find the radius of convergence. $\sum_{n=1}^{\infty} \frac{x^{3n}}{n 8^n}$ $r = 2$

5. Find the coefficient of x^5 in the Maclaurin series for $e^x \cos(2x)$. $1/120$

6. Find the the first three non-zero terms of Taylor series for $\cos 2x$ about $x = \pi/6$.

   $1/2 - \sqrt{3}(x - \pi/6) - (x - \pi/6)^2$

7. Find the Taylor series for $p(x) = x^3 + x^2 + x + 1$ about $x = 1$; i.e., write $p(x)$ as a polynomial in $x - 1$.

   $4 + 6(x-1) + 4(x-1)^2 + (x-1)^3$

8. Find the Maclaurin series for $\frac{2}{2-x}$. $1 + \frac{x}{2} + \frac{x^2}{2^2} + \frac{x^3}{2^3} + \cdots$

9. Use the Alternating Series Estimation Theorem to estimate the error in computing $\cos x$ by using $\cos x \approx 1 - \frac{x^2}{2} + \frac{x^4}{24}$ for $-1 \le x \le 1$.

   $|\text{error}| < \frac{1}{6!} = \frac{1}{720}$

Problems 10-15. Show your work for full credit.

10. Find the sum $\displaystyle\sum_{k=1}^{\infty} \frac{1}{k^2+k}$

Use partial fractions to find $\dfrac{1}{k^2+k} = \dfrac{1}{k} - \dfrac{1}{k+1}$

The partial sum $S_n = \frac{1}{1} - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} + \frac{1}{4} - \frac{1}{5} + \cdots + \frac{1}{n} - \frac{1}{n+1} = 1 - \frac{1}{n+1}$

Then $\displaystyle\lim_{n\to\infty} S_n = 1$ so the series converges to 1.

11. Evaluate the sum. $x + 2x^2 + 3x^3 + 4x^4 + \cdots = \displaystyle\sum_{n=1}^{\infty} nx^n$

Let $f(x) = x + 2x^2 + 3x^3 + 4x^4 + \cdots = \displaystyle\sum_{n=1}^{\infty} nx^n$. The $\frac{f(x)}{x} = 1 + 2x + 3x^2 + 4x^3 + \cdots = \sum_{n=1}^{\infty} nx^{n-1}$. Now $\int \frac{f(x)}{x} dx = C + x + x^2 + x^3 + x^4 + \cdots = C + \sum_{n=1}^{\infty} x^n$. We will compute this sum and take the derivative to find $\frac{f(x)}{x}$ so it doesn't matter what constant we choose. $C = 0$ is one choice, but it is easier to choose $C = 1$. With this choice of C, $1 + x + x^2 + x^3 + x^4 + \cdots = \frac{1}{1-x}$. So $\frac{f(x)}{x} = \frac{d}{dx}\frac{1}{1-x} = \frac{1}{(1-x)^2}$. Solving for $f(x)$ we get $f(x) = \dfrac{x}{(1-x)^2}$.

12. Evaluate the following limit: $\displaystyle\lim_{x\to 0} \frac{24 - 12x^2 + x^4 - 24\cos x}{x^6}$

Use the fact that $\cos x = 1 - \dfrac{x^2}{2!} + \dfrac{x^4}{4!} - \dfrac{x^6}{6!} + \dfrac{x^8}{8!} - \cdots$.

$$\lim_{x\to 0} \frac{24 - 12x^2 + x^4 - 24(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots)}{x^6} = \lim_{x\to 0} \frac{-24(-\frac{x^6}{6!} + \frac{x^8}{8!} - \cdots)}{x^6}$$

$$= \lim_{x\to 0} -24(-\frac{1}{6!} + \frac{x^2}{8!} - \cdots) = \frac{24}{6!} = \frac{1}{30}$$

13. Find the the first four non-zero terms of a power series expansion for the function $\dfrac{1}{\sqrt{1-x^2}}$ expanded about $x = 0$.

Use the Binomial Theorem. $\dfrac{1}{\sqrt{1-x^2}} = (1-x^2)^{-1/2}$

$$= 1 + \binom{-1/2}{1}(-x^2) + \binom{-1/2}{2}(-x^2)^2 + \binom{-1/2}{3}(-x^2)^3 + \cdots$$

$= 1 + \frac{-1/2}{1!}(-x^2) + \frac{(-1/2)(-3/2)}{2!}(-x^2)^2 + \frac{(-1/2)(-3/2)(-5/2)}{3!}(-x^2)^3 + \cdots$

$= 1 + \dfrac{1 \cdot x^2}{2 \cdot 1!} + \dfrac{1 \cdot 3 \cdot x^4}{2^2 \cdot 2!} + \dfrac{1 \cdot 3 \cdot 5 \cdot x^6}{2^3 \cdot 3!} + \cdots$

14. Find the interval of convergence. $\displaystyle\sum_{n=1}^{\infty} \frac{(2x-1)^n}{n^2 2^n}$

    Use the Ratio Test to find the radius of convergence. $\displaystyle\lim_{n\to\infty} \frac{|2x-1|n^2}{2(n+1)^2} = \frac{|2x-1|}{2}$.

    So the series converges for $|2x-1| < 2 \Rightarrow -2 < 2x-1 < 2 \Rightarrow -1/2 < x < 3/2$.

    At the endpoint $x = -1/2$ the series is $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$

    At the endpoint $x = 3/2$ the series is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$

    So the series converges at each endpoint and the interval of convergence is $[-1/2, 3/2]$.

15. Show the Ratio Test gives no information if the limiting ratio equals 1; i.e., give examples of two series whose limiting ratios are both 1 so that one converges and the other diverges.

    The harmonic series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n}$ diverges and the p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges. In each case the limiting ratio is 1.